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QUELQUES POINTS SUR LE FAISCEAU DES FONCTIONS J-HOLOMORPHES

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Soit (M, J) une variété presque complexe. Il existe une généralisation naturelle du faisceau des fonctions holomorphes sur M, définie par

$$\mathcal{O}_M^J(U) := \{f \in C^\infty(U, \mathbb{C}) \mid df \circ J = i\, df\}.$$

Ces fonctions, que nous appelons J-holomorphes, sont solutions d'un système elliptique du premier ordre surdéterminé. Kruglikov a démontré dans [14] qu'il n'existe pas de solutions locales non constantes pour un J générique; on peut donc s'attendre à ce que $\mathcal{O}_M^J$ ne soit intéressant que pour des "J spéciaux". À notre connaissance, le faisceau $\mathcal{O}_M^J$ a été considéré pour la première fois par Spencer en 1955 ([26], p. 20). Le problème d'existence de fonctions J-holomorphes avait été implicitement étudié auparavant par L. Nirenberg dans le cadre de son théorème de Frobenius complexe [21]. En 1986, O. Muškarov a donné un critère géométrique local pour l'existence locale de fonctions J-holomorphes ([19], Thm. 2.4) et à définie une notion de structure presque complex de "type constant" sur lesquel l'existence de fonctions J-holomorphe est facile à vérifier. Le problème d'existence a également été considéré par C. Han et H. Kim dans [9]. Le problème de la classification des germes de fonctions J-holomorphes a été étudié par Dimiev [7] en 1983. Dimiev a donné une classification complète pour le cas réel analytique en dimensions 4 et 6. Sa classification repose entièrement sur le comportement dimensionnel d'un seul objet, à savoir la clôture involutive de la distribution complexe associée à $TM^{1,0}$. Des développements plus implicites de la théorie des fonctions J-holomorphes ont été obtenus par Aziz El Kacimi-Alaoui dans les années 1990, comme corollaire à sa cohomologie de Dolbeault basique $H_{\mathrm{B}}^{p,q}$ [13]. Plus récemment, des progrès implicites ont été réalisés par Cirici-Wilson à travers leur cohomologie de Dolbeault généralisée $H_{\mathrm{Dolb}}^{p,q}(M)$ ([5], 2021), et par Cahen-Gutt-Gutt (CGG) à travers leur cohomologie de Dolbeault transverse que nous dénotons aussi par $H_B^{p,q}(M)$ ([4], 2024).

Ce mémoire est divisée en deux parties. Dans la première, nous nous intéressons aux comportements locaux de $\mathcal{O}_M^J$ pour J lisse. La section 2.1. rappelle le matériel nécessaire pour définir et donner un sens à la clôture involutive de la distribution complexe $TM^{1,0}$. En section 2.2., nous définissons la *coquille involutive* d'une structure presque complexe J, qui est une distribution réel noté $\mathcal{D}_J^\infty$, et expliquons comment elle obstrue (et parfois implique) l'existence de fonctions J-holomorphes. La distribution réelle $\mathcal{D}_J^\infty$ a également été définie dans CGG [4], bien que son lien avec les fonctions J-holomorphes n'y ait pas été étudié. À la section 3, nous utilisons $\mathcal{D}_J^\infty$ pour donner une classification des germes de fonctions J-holomorphes pour les structures presque complexes "de type constant" (voir Théorème 4). Notre approche est inspirée de l'article de Dimiev [7]. Dans la deuxième partie, nous étudions comment $\mathcal{O}_M^J$ interagit avec la géométrie (complexe) globale de (M, J).

Nous introduisons le phénomène que nous voulons comprendre dans la seconde partie par un exemple. Sur le tore de Kodaira-Thurston KT (défini en Exemple 4), il y a une structure presque complexe J non-intégrable et invariante à gauche canonique. Nous observons qu'il y a une "géométrie complex" intéressante sur (KT, J) bien que J soit non-intégrable. D'abord, l'image du tenseur de Nijenhuis de J est une distribution régulière involutive D. Les feuilles de cette distribution sont localement définies par des fonctions J-holomorphes et peuvent donc être vues comme des "sous-variétés J-holomorphes" (l'analogue des sous-variétés analytiques). Comme en géométrie complexe classique, ces sous-variétés J-holomorphes se "relèvent" en un fibré en droites J-holomorphe L, que nous comprenons comme un élément de $H^1(KT, \mathcal{O}_{KT}^{J,*})$. Nous expliquons maintenant quelles données géométriques est encodée première classe de Chern $c_1(L)$. L'espace des feuilles $\pi : KT \to KT/D =: T^2$ est une surface de Riemann de genre 1. Soit $[\mathrm{pt}] \in H^1(T^2, \mathcal{O}_{T^2}^*)$ la classe d'un point. La fonctorialité de c_1 implique que $c_1(L) = \pi^* c_1([\mathrm{pt}])$ engendre le groupe de cohomologie de Dolbeault basique en bidegré $(1,1)$ de KT (c.-à-d. $H_B^{1,1}(KT)$, voir Défininition 11). Dans ce cas particulier, $c_1(L)$ peut être vu comme générant $H^2(T^2, \mathbb{Z}) \cap H_B^{1,1}(KT)$ (voir Exemple 8 pour les détails). Ceci doit être comparé avec le théorème de Lefschetz sur les classes de type $(1,1)$ pour les variétés kählériennes.

Question 1. Pour quelles variétés presque kählériennes existe-t-il un "théorème de Lefschetz transverse" pour les classes $(1,1)$?

Les groupes de cohomologie de Dolbeault transverse de KT ont également des interprétations naturelles en termes de $\mathcal{O}_{KT}^J$ via un "isomorphisme de Dolbeault basique". Plus précisément, on peut définir un complexe de faisceaux de "formes holomorphes basiques" $\Omega_B^\bullet$ (voir Définition 10) sur KT. La cohomologie de faisceau de $\Omega_B^\bullet$ est alors étroitement liée à $H_B^{p,q}(KT)$ (voir Corollaire 4 pour un énoncé précis). Cela mène à la question suivante :

Question 2. Sur quelles variétés presque complexes peut-on retrouver un isomorphisme "de Dolbeault basique" ?

Dans le cas du tore de Kodaira-Thurston, la cohomologie de Dolbeault récemment introduite par Cirici-Wilson [5], notée $H_{\mathrm{Dolb}}^{p,q}$, est aussi étroitement reliée à la cohomologie de Dolbeault transverse de Cahen-Gutt-Gutt. En effet, Cahen-Gutt-Gutt ont observé que, pour toute variété presque complexe, il existe une application naturelle $H_B^{p,q}(M) \to H_{\mathrm{Dolb}}^{p,q}(M)$ induisant un isomorphisme naturel $H_B^{0,q}(M) \cong H_{\mathrm{Dolb}}^{0,q}(M)$ pour tout $q \geq 0$. Dans le cas particulier de KT, cette application naturelle est en fait injective en tout degré. Notons qu'on ne peut espérer un isomorphisme, comme on le voit à partir des diamants de Hodge suivants :

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \qquad\qquad \begin{array}{ccccc} & & 1 & & \\ & 2 & & 1 & \\ 0 & & 4 & & 0 \\ & 1 & & 2 & \\ & & 1 & & \end{array}$$

$$\dim H_{J-\mathrm{trans}}^{p,q}(KT) \qquad\qquad \dim\ {}^L H_{\mathrm{Dolb}}^{p,q}(KT)$$

Le diamand de gauche est la cohomologie de Dolbeault transverse de KT [4], Thm. 4.5., et le diamand de droite donne une borne inférieure pour le diamant de Hodge de la cohomologie de Dolbeault de KT [5], Ex. 5.16.

Question 3. Sur quelles variétés presque kählériennes l'application naturelle $H_B^{p,q}(M) \to H_{\mathrm{Dolb}}^{p,q}(M)$ est-elle injective ?

Une question connexe, à laquelle nous n'avons pas de réponse, est :

Question 4. Lorsque l'application $H_B^{p,q}(M) \to H_{\mathrm{Dolb}}^{p,q}(M)$ est injective, quelle "propriété géométrique" de M les générateurs supplémentaires de $H_{\mathrm{Dolb}}^{p,q}(M)$ capturent-ils ?

Pour répondre aux questions 1, 2 et 3, nous introduisons une notion de variétés *transversalement kählériennes* et *minimalement* transversalement kählériennes en section 4.4. (voir définition 13). Ce sont des variétés presque kählériennes avec des hypothèse supplèmentaire sur $\mathcal{D}_J^\infty$. Ces "hypothèses supplémentaires" sont imposées afin que la théorie harmonique d'El Kacimi sur les feuilletages kählériens minimaux [13] s'applique aux variétés transversalement kählériennes minimales (c'est la "théorie harmonique" que nous utilisons pour démontrer une version du théorème de Lefschetz sur les classes $(1,1)$). Le travail d'El Kacimi que nous utilisons est exposé en sections 4.1., 4.2. et 4.3.

À la section 4.5., nous montrons que les variétés transversalement kählériennes minimales forment une classe de variétés répondant à la question 3 (voir Proposition 9). Nous abordons la question 2 en section 4.6. et la question 3 en section 5. Pour ces deux questions, nous avons dû imposer la condition supplémentaire que les feuilles du feuilletage induit par $\mathcal{D}_J^\infty$ soient compactes (voir Corollaire 4 et Théorème 9). Nous ne savons pas s'il est possible de supprimer ou d'affaiblir cette hypothèse; c'est pourquoi nous ne l'avons pas inclu dans la définitions des variétés transversalement kählériennes.

Nous concluons l'introduction par une remarque qui motive la notion de variétés transversalement kählériennes d'un autre point de vue. Il existe dans la théorie symplectique des 4-variétés une conjecture de Tian-Jun Li :

Conjecture I (voir [17]). La variété sous-jacente à toute variété symplectique (M, ω) de dimension 4 avec $c_1(M) = 0$ est soit kählérienne, soit un fibré en tores T^2 au-dessus de T^2.

Comme les variétés transversalement kählériennes sont en particulier symplectiques, la conjecture de Tian-Jun Li impliquerait :

Conjecture II. La variété sous-jacente à toute variété transversalement kählérienne (M, J, g) de dimension réelle 4 avec $c_1(M) = 0$ est soit kählérienne, soit un fibré en tores T^2 au-dessus de T^2.

La Conjecture II pourrait être plus facile à vérifier, puisque la donnée d'une structure transversalement kählérienne est a priori beaucoup plus restrictives que celles d'une structure symplectique. La Conjecture I serait alors équivalente au problème de trouver un $J \in \mathcal{J}_{\mathrm{comp}}(M, \omega)$ qui rende (M, ω) transversalement kählérienne.

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1 INTRODUCTION

Let (M, J) be an almost complex manifold. There is a natural generalization of the sheaf of holomorphic functions on M, defined by

$$\mathcal{O}_M^J(U) := \{f \in C^\infty(U, \mathbb{C}) \mid df \circ J = i\, df\}.$$

These functions, which we call J-holomorphic, are solutions to an overdetermined first-order elliptic system. Kruglikov proved in [14] that there is no non-constant local solutions for generic J, thus one can expect $\mathcal{O}_M^J$ to be interesting only for "special J's". To our knowledge, the sheaf $\mathcal{O}_M^J$ was first considered by Spencer in 1955 ([26] p.20). The existence problem of J-holomorphic functions was implicitly considered before by L. Nirenberg as a part of his complex Frobenius theorem [21]. In 1986 O. Muškarov gave a neat local geometric criterion for the local existence of J-holomorphic functions ([19] Theorem 2.4.) and defined families of well behaved almost complex structure with respect to J-holomorphic functions which he called "of constant type". The existence problem was also considered by C. Han and H. Kim in [9]. The problem of classifying the germs of J-holomorphic functions was considered by Dimiev [7] in 1983. He gave a complete classification in the real analytic setting in dimensions 4 and 6. His classification is entirely based on the dimensional behavior of a single object, namely the involutive closure of the complex distribution associated to $TM^{1,0}$. More implicit development on the theory of J-holomorphic functions has been achieved by Aziz El Kacimi-Alaoui in the 90s as a by-product of his basic Dolbeault cohomology, denoted $H_{\mathrm{B}}^{p,q}$ [13]. More recently, implicit progress was achieved by Cirici-Wilson as a by-product of their generalized Dolbeault cohomology denoted $H_{\mathrm{Dolb}}^{p,q}(M)$ ([5], 2021) and by Cahen-Gutt-Gutt (CGG) as a by-product of their transverse Dolbeault cohomology, which we also denote $H_B^{p,q}(M)$ ([4], 2024).

The thesis is divided in two parts. In the first part, we are interested in the local behaviours of $\mathcal{O}_M^J$ for smooth J. In section 2.1., we recall the material needed to define and make sens of the involutive closure of the complex distribution $TM^{1,0}$. In section 2.2., we define the *involutive hull* of an almost complex structure J , denoted $\mathcal{D}_J^\infty$, and explain how it obstruct (and sometime implies) the existence of J-holomorphic functions. The real distribution $\mathcal{D}_J^\infty$ was also defined in CGG [4] although its relation to J-holomorphic functions was not investigated there. In section 3, we use $\mathcal{D}_J^\infty$ to give a classification of the germs of J-holomorphic functions for the almost complex structure of "constant type" (see Theorem 4). Our approach was inspired by Dimiev's paper [7]. In the second part, we study how $\mathcal{O}_M^J$ interacts with the global (complex) geometry of (M, J).

We introduce the phenomenon we want to understand in the second part with an example. Consider the canonical left-invariant *non-integrable* almost complex structure J on the Kodaira-Thurston torus KT (defined in Example 4). As it will turn out, the image of the Nijenhuis tensor of KT is an involutive regular distribution D. The leaves of this distribution are locally cut-out by J-holomorphic functions and thus can be seen as "J-holomorphic subvarieties" (the analogue of analytic subvarieties). As in classical complex geometry, these J-holomorphic subvarieties "lift" to a J-holomorphic line bundle L which we understand as an element of $H^1(KT, \mathcal{O}_{KT}^{J,*})$. We now explain what geometric data the first Chern class $c_1(L)$ is encoding. The leaf space $\pi : KT \to KT/D =: T^2$ is a genus 1 Riemann surface. Let $[\mathrm{pt}] \in H^1(T^2, \mathcal{O}_{T^2}^*)$ be the class of a point. Functoriality of c_1 implies that $c_1(L) = \pi^* c_1([\mathrm{pt}])$ generate the $(1,1)$-Basic Dolbeault cohomology group of KT (i.e. $H_B^{1,1}(KT)$ see Definition11). In this specific instance, $c_1(L)$ can be seen to generate $H^2(T^2, \mathbb{Z}) \cap H_B^{1,1}(KT)$ (see Example 8 for the details). This should be compared with the Lefschetz theorem on $(1,1)$ classes for Kähler manifolds:

Question 1. For which almost Kähler manifolds does a "transverse Lefschetz theorem" for (1,1)-classes hold?

The transverse Dolbeault cohomology groups of KT also have natural interpretations in terms of $\mathcal{O}_{KT}^J$ via a "basic Dolbeault isomorphism". More precisely, one can define a sheaf complex of "basic holomorphic forms" $\Omega_B^\bullet$ (see def. 10) on KT. The sheaf cohomology of $\Omega_B^\bullet$ is then closely related to $H_B^{p,q}(KT)$ (see Corollary 4 for a precise statement). This leads to the

Question 2. On which almost complex manifolds can one recover a "Basic Dolbeault" isomorphism ?

On the Kodaira-Thurston Torus, the Dolbeault cohomology of Cirici-Wilson [5], denoted $H_{\mathrm{Dol}}^{p,q}(KT)$, is also closely related to the transverse Dolbeault cohomology of Cahen-Gutt-Gutt [4]. First, Cahen-Gutt-Gutt observed that for any almost complex manifold, there is a natural map $H_B^{p,q}(M) \to H_{\mathrm{Dolb}}^{p,q}(M)$ inducing a natural isomorphism $H_B^{0,q}(M) \cong H_{\mathrm{Dol}}^{0,q}(M)$ for all $q \geq 0$. On KT, this natural map is actually an injection in all degrees. Note that one cannot hope for an isomorphism, this is easily seen from the following Hodge diamonds on KT:

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 2 & & 1 & \\ 0 & & 4 & & 0 \\ & 1 & & 2 & \\ & & 1 & & \end{array}$$

$$\dim H^{p,q}_{J-\mathrm{trans}}(KT) \qquad \dim {}^{L}H^{p,q}_{\mathrm{Dolb}}(KT)$$

on the left is the transverse Dolbeault cohomology of KT [4], thm. 4.5., and the diamond on the right gives a lower-bound for the Hodge diamond of the Dolbeault cohomology of KT [5], Example. 5.16.

Question 3. On which almost Kähler manifold is the natural map $H^{p,q}_B(M) \to H^{p,q}_{\mathrm{Dol}}(M)$ injective ?

A related question for which we have no answer is

Question 4. Whenever the map $H^{p,q}_{J-\mathrm{trans}}(M) \to H^{p,q}_{\mathrm{Dolb}}(M)$ is injective, what "geometric feature" of M the additional generator in $H^{p,q}_{\mathrm{Dolb}}(M)$ are capturing?

To answer question 1, 2 and 3, we introduce a notion of *transversaly Kähler* manifolds and *minimal* transversely Kähler manifolds in section 4.4. (see def. 13). These are almost Kähler manifold with extra hypothesis. The "extra hypothesis" are imposed so that the harmonic theory of EL. Kacimi on minimal Kähler foliations [13] applies to minimal transversely Kähler manifolds (this is the "Harmonic theory" input that we use to prove a version of Lefschetz theorem on (1,1)-classes). The work of EL. Kacimi that we use is exposed in section 4.1., 4.2. and 4.3.

In section 4.5., we prove that minimal transversaly Kähler manifold gives a class of manifolds answering question 3 (see Proposition 9). We tackle question 2 in section 4.6. and question 3 in section 5. For these two questions, we needed to impose the additional condition that leaves of the foliation induced by $\mathcal{D}^{\infty}_J$ be compact (see Corollary 4 and Theorem 9). We are unsure whether it is possible to remove or weaken this hypothese so we excluded it from the definition of transversaly Kähler manifold.

We conclude the introduction on a remark which motivates the notion of transversely Kähler manifolds from another viewpoint. There is in the symplectic theory of 4-manifolds, a folklore conjecture of Tian-Jun Li:

Conjecture I (see [17]). The underlying smooth manifold of any symplectic manifold (M, ω) of dimension 4 with $c_1(M) = 0$ is either a Kähler manifold or a T^2-bundle over T^2.

Since transversely Kähler manifolds are in particular symplectic, the conjecture of Tian-Jun Li would imply:

Conjecture II. The underlying smooth manifold of any transversely Kähler manifold (M, J, g) of (real) dimension 4 with $c_1(M) = 0$ is either a Kähler manifold or a T^2-bundle over T^2.

Conjecture II may be easier to check, since the data of a transversely Kähler structure is a priori much more restrictive than the data of a symplectic structure. Then conjecture I would be equivalent to the problem of finding a J in $\mathcal{J}_{\mathrm{comp}}(M, \omega)$ which makes (M, ω) transversely Kähler.

We begin by recalling conventions and basic results in the theory of distributions and Pfaffian systems. This culminates in the definition of the involutive hull of an almost complex structure (see Definition. 3), which will be the main object of interest for our study of J-holomorphic functions.

2.1 Distributions, Pfaffian Systems, and Derived Flags.

The purpose of this section is to define the basic objects we want to manipulate. The material on distributions can be found in [7] §1, [4] §1 and [16] §19. The material on Pfaffian systems can be found in [9] §1, together with [10] §3. Taking the "closed" limit does not seem to be a standard operation, but there is nothing complicated about it.

Let M be a smooth manifold (second countable and Hausdorff). A *(generalized) distribution* is a $C^\infty(M,\mathbb{R})$-submodule of $\Gamma(TM)$ (Γ will be used to denote smooth global sections of a sheaf or vector bundle). Because M is Hausdorff and paracompact, it is not hard to see that there is an equivalence of categories between the subsheaves of $C^\infty(M,\mathbb{R})$-modules of TM and generalized distributions. When a distribution is also a subbundle of TM, we call it *regular.* Given a distribution $\mathcal{D}$ of TM we define its fibre $\mathcal{D}_x$ at a point $x \in M$ to be the $\mathbb{R}$-vector space

$$\mathcal{D}_x := \{Y \in T_xM \mid \exists\, X \in \mathcal{D} \text{ with } X_x = Y\,\}.$$

We then define the *rank* of $\mathcal{D}$ at x to be rank $\mathcal{D}_x$. The rank need not be constant when $\mathcal{D}$ is not regular. However, we always have the following property :

Lemma 1. Let M^n be a smooth manifold of dimension n and $\mathcal{D}$ a distribution on M. The map $x \mapsto \operatorname{rank} \mathcal{D}_x$ is lower semi-continuous. Equivalently, for each integer $k = 1, \ldots, n$, the set $\{x \in M \;:\; \operatorname{rank} \mathcal{D}_x \geq k\}$ is open.

Proof. Let $x \in M$ and $k := \operatorname{rank}\mathcal{D}_x$. By definition of k, there exists k vector fields $X^1, ..., X^k$ that are linearly independent at x. Since being linearly independent is an open condition, the X^i's are also linearly independent in a neighbourhood of x. Hence $\operatorname{rank}\mathcal{D}_y \geq k$ for each y this neighbourhood. □

We say that $\mathcal{D}$ is *involutive* if $[\mathcal{D},\mathcal{D}] \subset \mathcal{D}$. Any distribution has an associated derived flag defined inductively as follows:

$$\mathcal{D}^{(0)} := \mathcal{D}, \quad \mathcal{D}^{(k+1)} := \mathcal{D}^{(k)} + [\mathcal{D}^{(k)},\mathcal{D}^{(k)}] \quad \text{and} \quad \mathcal{D}^\infty := \bigoplus_{n\in\mathbb{N}} \mathcal{D}^{(n)}.$$

We call $\mathcal{D}^\infty$ the *involutive limit* of the derived flag. It is the smallest involutive distribution containing $\mathcal{D}$ (in the sense that any other involutive distribution $\mathcal{D}'$ containing $\mathcal{D}$ is contained in $\mathcal{D}^\infty$). Finally, note that all the above discussion extend to *complex* distributions, defined as $C^\infty(M,\mathbb{C})$ submodules of $\Gamma(TM_\mathbb{C})$ (where $TM_\mathbb{C} := TM \otimes_\mathbb{R} \mathbb{C}$).

A *Pfaffian system* is a $C^\infty(M,\mathbb{R})$-submodule of the space of smooth 1-forms $\Omega^1(M,\mathbb{R})$. A Pfaffian system $\mathcal{I}$ is *d-closed* if $d\mathcal{I} \subset \mathcal{I} \wedge \Omega^1(M)$ (in the sense that the exterior derivative of any one form $\sigma \in \mathcal{I}$ can be expressed locally as a linear combination $d\sigma = \sum_i \sigma_i \wedge \xi_i$ with $\sigma_i \in \mathcal{I}$ and

$\xi_i \in \Omega^1(M)$). The *rank* and *fibre* at a point are defined analogously to the case of distributions. The rank is again lower semi-continuous. $\mathcal{I}$ is said to be *regular* if it is a subbundle of T^*M. The derived flag of $\mathcal{I}$ is given inductively as follows:

$$\mathcal{I}^{(0)} := \mathcal{I}, \quad \mathcal{I}^{(k+1)} := \{\sigma \in \mathcal{I}^{(k)} | d\sigma \in \mathcal{I}^{(k)} \wedge \Omega^1(M)\} \quad \text{and} \quad \mathcal{I}^\infty := \cap_{n\in\mathbb{N}} \mathcal{I}^{(n)}$$

We call $\mathcal{I}^\infty$ the *closed limit* of the derived flag of $\mathcal{I}$. It is the largest closed Pfaffian system contained in $\mathcal{I}$.

We now turn to the relations between Pfaffian systems and distributions. If $\mathcal{D}$ is a distribution on M, we denote by $\mathcal{D}^\perp$ the Pfaffian system of 1-forms vanishing on $\mathcal{D}$. Given a Pfaffian system $\mathcal{I}$, we let $\mathcal{I}^\perp$ be the vanishing set of $\mathcal{I}$: $\mathcal{I}^\perp := \{X \in \mathfrak{X}(M) \mid \sigma(X) \equiv 0 \,\forall \sigma \in \mathcal{I}\}$. It is a distribution of M. In the case where $\mathcal{D}$ is regular (resp. $\mathcal{I}$ is regular), the relation $(\mathcal{D}^\perp)^\perp = \mathcal{D}$ (resp. $(\mathcal{I}^\perp)^\perp = \mathcal{I}$) holds. In the non-regular case, we only have $\mathcal{D} \subset (\mathcal{D}^\perp)^\perp$ and $\mathcal{I} \subset (\mathcal{I}^\perp)^\perp$, as the following example shows:

Example 1. Let $M = \mathbb{R}$, the pairing between one-form and vector fields is simply multiplication of functions: $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$. Let φ_1 (resp. φ_2) be a smooth function such that $\varphi_1(x) = 0$ if $x \leq 0$ (resp. $\varphi_2(x) = 0$ if $x \geq 0$), φ_1 and φ_2 vanish to infinite order at 0 and $\varphi_1(x) > 0$ if $x > 0$ (resp. $\varphi_2(x) > 0$ if $x < 0$). Let $\mathcal{I} := \langle \varphi_1, \varphi_2 \rangle_{C^\infty(\mathbb{R})}$ be the submodule generated by φ_1 and φ_2. $\mathcal{I} \neq C^\infty(\mathbb{R})$ since any 1-form $\sigma \in \mathcal{I}$ must vanish to infinite order at 0. However $(\mathcal{I}^\perp)^\perp = I(0) = C^\infty(\mathbb{R})$.

If follows from the invariant formula

$$d\sigma(X, Y) + \sigma([X, Y]) = X(\sigma(Y)) - Y(\sigma(X)),$$

where $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Omega^1(M)$, that if $\mathcal{I}$ is closed (resp. $\mathcal{D}$ is involutive) then $\mathcal{I}^\perp$ is involutive (resp. $\mathcal{D}^\perp$ is closed). Now let $\mathcal{I}$ be a Pfaffian system and $\mathcal{D} := \mathcal{I}^\perp$. We wish to compare their derived flags. If for each $k \in \mathbb{N}$, $\mathcal{I}^{(k)}$ is regular, then the derived flag has finite length (there exists $n \in \mathbb{N}$ such that $\mathcal{I}^{(n+k)} = \mathcal{I}^{(n)}$ for all $k \in \mathbb{N}$) and $\mathcal{I}^{(k)} = (\mathcal{D}^{(k)})^\perp$ for each $k \in \mathbb{N}$. Note that even if $\mathcal{I}$ is regular, its derived flag may not be regular and the above correspondence of flags does not hold in general. However, the flag of a regular Pfaffian is always well behaved "in the limit":

Lemma 2. Let $\mathcal{I}$ be a regular Pfaffian system and $\mathcal{D} := \mathcal{I}^\perp$. Then $(\mathcal{D}^\infty)^\perp = \mathcal{I}^\infty$.

Proof. Since $\mathcal{D}^\infty$ is involutive and contains $\mathcal{I}^\perp$, $(\mathcal{D}^\infty)^\perp$ is closed and is contained in $(\mathcal{I}^\perp)^\perp = \mathcal{I}$. Since $\mathcal{I}^\infty$ is the maximal closed Pfaffian system contained in $\mathcal{I}$, we conclude that $(\mathcal{D}^\infty)^\perp \subset \mathcal{I}^\infty$. Now $(\mathcal{I}^\infty)^\perp$ contains $\mathcal{I}^\perp = \mathcal{D}$ and is involutive, so we must have $\mathcal{D}^\infty \subset (\mathcal{I}^\infty)^\perp$. Then $\mathcal{I}^\infty \subset ((\mathcal{I}^\infty)^\perp)^\perp \subset (\mathcal{D}^\infty)^\perp$. $\square$

2.2 Local obstruction and existence of J-holomorphic functions

Let (M^{2n}, J) be an almost complex manifold and let $p \in M$. We are interested in the number of functionally independent[1] J-holomorphic functions at p. Following Muškarov [19], we have the following definition:

[1]Two J-holomorphic function f, g are functionally independent at p if df_p and dg_p are linearly independent as $\mathbb{C}$-linear map $(T_pM, J_p) \to (\mathbb{C}, i)$.

Definition 1. Let (M^{2n}, J) be an almost complex manifold and let $m(x)$ denote the maximal number of functionally independent J-holomorphic functions at $x \in M$. We say that (M, J) is of *type* m at $p \in M$ if $m(x) \equiv m$ in a neighbourhood of p. We say (M, J) is of *(constant) type* m if $m(x) \equiv m$ everywhere.

Example 2 (see [19], §4 and §5)**.** Left invariant almost complex structures on homogeneous spaces are of constant type. Likewise, the almost complex structure of a nearly Kähler manifold is of constant type.

Remark 1. Almost complex structure of constant type are the "easiest to study". Lemma 3 shows that the increase in difficulty in studying almost complex structures of non-constant type is comparable to the increase in difficulty when passing from the study of regular Pfaffian systems to non-regular one (see §2.1 for the definition of Pfaffian system).

We will show that the derived flag $(TM^{1,0})^{(k)}$ of the distribution $TM^{1,0}$ in $TM_{\mathbb{C}}$ controls the existence of J-holomorphic functions to some extent. For reasons that will become clear in Theorem 3 and 4, the "real part" (defined below) of $(TM^{1,0})^{\infty}$ interacts more directly with J-holomorphic functions than $(TM^{1,0})^{\infty}$ itself, we will therefore state our theorems in terms of it.

Definition 2 (compare with [19], p.286)**.** Let (M^{2n}, J) be an almost complex manifold and let $V \subset TM_{\mathbb{C}}$ be a complex distribution containing $TM^{1,0}$. We define the associated real distribution $V_{\mathbb{R}}$ to be the real distribution defined by $V_{\mathbb{R}} := \{X \in \Gamma(TM) \,:\, X + iJX \in \Gamma(V^{0,1})\}$.

Remark 2. If $\dim M = 2n$, it is straightforward to see that, at any point, $\operatorname{rank}_{\mathbb{R}} V_{\mathbb{R}} = 2 \operatorname{rank}_{\mathbb{C}} V - 2n$

Definition 3. Given an almost complex manifold (M, J) we define the *involutive hull*[2] *of* J to be $\mathcal{D}_J^{\infty} := ((TM^{1,0})^{\infty})_{\mathbb{R}}$.

Now recall the following lemma of Muškarov, which relies on Nirenberg's complex Frobenius theorem:

Lemma 3 ([19], lemma 2.2.)**.** The local existence of k independent J-holomorphic functions on (M^{2n}, J) is equivalent to the local existence of a d-closed subbundle of $T^*M^{1,0}$ (or equivalently of $T^*M^{0,1}$) with complex fibre of dimension k.

proof sketch. Let $F \subset T^*M^{1,0}$ be a closed subbundle of constant complex rank k (real rank $2k$). Notice that if we can find local coordinates $\varphi = (x^1, ..., x^{2n})$ around each point of M such that $dx^m + idx^{m+n}$ $(m = 1, .., k)$ locally span F, we will be done. Indeed, letting $f_m := x^m + ix^{m+n}$, we get that the image of df_m belongs to $F \subset T^*M^{1,0}$, so that $\bar{\partial}_J f_m = (df_m)^{0,1} = 0$ and f_m is J-holomorphic. Clearly the f_m's are functionally independent. The existence of such a coordinate chart φ is the hard part and follows directly from L. Nirenberg's complex Frobenius theorem ([21], theorem 1). (This is where the d-closed hypothesis is used.) □

[2] We will shortly see that this coincides with the distribution $\mathcal{D}_J^{\infty}$ introduced in Cohen-Gutt-Gutt's [4], Proposition 1.4.

Remark 3. The case $k = n$ is equivalent to the statement of the Newlander-Nirenberg's theorem [20].

Muškarov's lemma is stated in terms of the existence of a local d-closed subbundle of $T^*M^{0,1}$. Our observation is that whenever this bundle exists, it coincides with $(T^*M^{0,1})^\infty$ locally.

Proposition 1. Let (M^{2n}, J) be an almost complex manifold. The Pfaffian system $(T^*M^{0,1})^\infty$ has constant (complex) rank $0 \leq m \leq n$ in a neighbourhood of $p \in M$ if and only if (M, J) has type m at p.

Proof. Begin by assuming that $(T^*M^{0,1})^\infty$ has constant rank m in a neighbourhood of p. Lemma 3 implies that there exists m functionally independent J-holomorphic functions at p and there can't be more by maximality of $(T^*M^{0,1})^\infty$, thus (M, J) has type m at p.

Now assume that (M, J) has type m at p. By lemma 3 there exists an open subset $p \in U \subset M$ and a d-closed subbundle $F \subset T^*M^{0,1}|_U$ of (complex) rank m. Observe that (M, J) has type m at each point of U. We want to show that $F = (T^*M^{0,1})^\infty|_U$. Note that if $n = m$, then $F = (T^*M^{0,1})^\infty$ is immediate. Hence we assume $n \neq m$.

Suppose for contradiction that $F \subsetneq (T^*M^{0,1})^\infty|_U$ (i.e. F is a proper subset of $(T^*M^{0,1})^\infty$). There must be $q_1 \in U$ such that $F_{q_1} \subsetneq (T^*M^{0,1})^\infty_{q_1}$ and in particular,

$$m = \text{rank } F_{q_1} < \text{rank } (T^*M^{0,1})^\infty_{q_1} =: m_1. \quad \text{(where the rank is taken over } \mathbb{C})$$

Since $x \mapsto \text{rank } (T^*M^{0,1})^\infty_x$ is a lower semi-continuous map (cf. lemma 1), $\text{rank } (T^*M^{0,1})^\infty_x \geq m_1$ for all x in a neighbourhood $U_1 \subset U$ of q_1. If $\text{rank } (T^*M^{0,1})^\infty|_{U_1} \equiv m_1$, then (M, J) has type $m_1 > m$ on the points of U_1, contradicting the hypotheses. Hence, there is $q_2 \in U_1$ such that

$$m_1 = \text{rank } (T^*M^{0,1})^\infty_{q_1} < (T^*M^{0,1})^\infty_{q_2} =: m_2.$$

Again by lower semi-continuity, we see that there is a neighbourhood $y_2 \in U_2 \subset U_1$ such that $\text{rank } ((T^*M^{0,1})^\infty)_x|_{U_2} \geq m_2$. If $\text{rank } (T^*M^{0,1})^\infty|_{U_2} \equiv m_2$, then (M, J) has type $m_2 > m$ on $U_2 \subset U$, a contradiction. Hence there is a point $q_3 \in U_2$ such that $m_2 = \text{rank } T^*M^{1,0}_{q_2} < (T^*M^{0,1})^\infty_{q_3} =: m_3$.

We have just built a sequence $m_3 > m_2 > m_1$. Continuing this process inductively at most up to n, we end up with an integer k such that there is a point $q_k \in U_{k-1} \subset U$ such that $\text{rank}\,(T^*M^{0,1}_{q_k})^\infty = m_k = n$. Then, by lower semi-continuity of the rank, there is a neighbourhood $q_k \in U_k \subset U_{k-1}$ such that $\text{rank } (T^*M^{0,1})^\infty|_{U_k} \equiv n$. But then this implies that (M, J) has type n at some point of U, a contradiction. □

Corollary 1. Let (M^{2n}, J) be an almost complex manifold, if $(TM^{1,0})^\infty$ is regular in a neighbourhood of p, then (M, J) has type m at p, where m is defined by $2(n - m) = \text{rank}_{\mathbb{R}}(\mathcal{D}^\infty_J)_p$.

Proof. Observe that $\text{rank}_{\mathbb{C}}(TM^{1,0})^\infty = 2n - \text{rank}_{\mathbb{C}}(T^*M^{0,1})^\infty$ and that $\text{rank}_{\mathbb{R}}\, \mathcal{D}^\infty_J := 2(\text{rank}_{\mathbb{C}}(TM^{1,0})^\infty - n)$. Putting the two equality together yields the conclusion. □

The upshot is that when $\mathcal{D}_J^\infty$ is regular near a point, the existence problem of J-holomorphic functions at that point is understood. We do not know how to tackle the existence problem when $\mathcal{D}_J^\infty$ is non-regular in an arbitrary small neighbourhood of a point. In that case, at least, the Nijenhuis tensor (and more sharply $\mathcal{D}_J^\infty$ itself) gives a quantitative bound on the number of J-holomorphic functions (see Theorem 1 below).

Recall that the Newlander-Nirenberg theorem asserts that there exists locally n functionnaly independent J-holomorphic functions on (M^{2n}, J) iff $TM^{1,0}$ is involutive. This is equivalent to requiring $[X,Y]^{0,1} = 0$ for all $X, Y \in TM^{1,0}$, where the notation $X^{0,1}$ means the image of X under the projection $TM^{1,0} \oplus TM^{0,1} = TM_{\mathbb{C}} \to TM^{0,1}$. We then define the (complex) Nijenhuis tensor $N_J^{\mathbb{C}} : \wedge^2 TM^{1,0} \to TM^{0,1}$ to be $N_J^{\mathbb{C}}(X,Y) = [X,Y]^{0,1}$ which is precisely the obstruction to integrability. Upon noticing the isomorphisms (see [11], lemma 1.2.5.)

$$\begin{aligned} TM &\longrightarrow TM^{1,0} & \qquad TM &\longrightarrow TM^{0,1} \\ X &\mapsto X - iJX & X &\mapsto X + iJX \end{aligned} \tag{1}$$

the condition for $TM^{1,0}$ to be involutive also writes

$$[X - iJX, Y - iJY] + iJ[X - iJX, Y - iJY] = 0$$

for all $X, Y \in TM$. Simplifying this expression yield a condition for integrability in term of the (real) Nijenhuis tensor: $TM^{1,0}$ is involutive if and only if $N_J(X,Y) = 0$ for all $X, Y \in TM$, where

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY].$$

Given $X, Y \in TM$, a short computation shows that $N_J^{\mathbb{C}}$ and N_J are related by the equation

$$N_J^{\mathbb{C}}(X - iJX, Y - iJY) = N_J(X,Y) + iJN_J(X,Y). \tag{2}$$

Theorem 1. Let (M, J) be an almost complex manifold and $p \in M$ a point. Denote by k the number of functionally independent J-holomorphic function at p. Then[3]

$$2n - 2k \geq \limsup_{x \to p}(\mathrm{rank}_{\mathbb{R}}(\mathcal{D}_J^\infty)_x) \geq \limsup_{x \to p}(\mathrm{rank}_{\mathbb{R}}\,(N_J)_x)$$

The idea is that the maximal value of $\mathrm{rank}\,\mathcal{D}_J^\infty$ that persists in arbitrarily small neighbourhood of p obstructs the existence of J-holomorphic functions.

Proof. Suppose there are k functionally independent J-holomorphic functions defined in a neighbourhood U of p, Then by Lemma 3, there is a d-closed subbundle $F \subset T^*M^{0,1}|_U$ of rank k. Let $(TM^{1,0})^{(m)}$ be the derived flag of $TM^{1,0}|_U$ (c.f. Subsection 2.1). Since $T^*M^{0,1}$ is a regular Pfaffian system, Lemma 2 implies that $((TM^{1,0})^\infty)^\perp$ is the largest d-closed Pfaffian system contained in $T^*M^{0,1}$. It follows that $F \subset ((TM^{1,0})^\infty)^\perp$. In particular, $\mathrm{rank}\,F_x \leq \mathrm{rank}\,((TM^{1,0})^\infty)^\perp_x$ for each $x \in U$. Since $\mathrm{rank}\,F_x$ is a constant

[3]Recall that given a topological space (X, τ) and a function $f : X \to \mathbb{R}$, $\limsup_{x \to p} f(x)$ is defined as $\inf\limits_{U \in \tau \,:\, p \in U} \sup\limits_{x \in U} f(x)$

function on U and $\operatorname{rank}((TM^{1,0})^\infty)^\perp_x$ takes a finite number of values, we also have

$$k = \operatorname{rank} F_x = \liminf_{x\to p}(\operatorname{rank} F_x) \le \liminf_{x\to p}(\operatorname{rank}\,((TM^{1,0})^\infty)^\perp_x) \qquad (\dagger)$$

The conclusion follows from three facts. First, observe that $\operatorname{rank}((TM^{1,0})^\infty)^\perp_x = 2n - \operatorname{rank}((TM^{1,0})^\infty)_x$. Second, we have $TM^{1,0} \oplus \operatorname{Im}(N_J^{\mathbb{C}}) = (TM^{1,0})^{(1)} \subset (TM^{1,0})^\infty$ by definition of $(TM^{1,0})^\infty$. Third, it follows from Equation (2) that $\operatorname{rank}_{\mathbb{R}} N_J = 2\operatorname{rank}(TM^{1,0} \oplus \mathcal{I}m(N_J^{\mathbb{C}})) - 2n$. Then we see:

$$\begin{aligned}\liminf_{x\to p}(\operatorname{rank}\,((TM^{1,0})^\infty)^\perp) &= 2n - \limsup_{x\to p}\,(\operatorname{rank}\,(TM^{1,0})^\infty)_x\\ &\le 2n - \limsup_{x\to p}\,(\operatorname{rank} TM^{1,0}\oplus\operatorname{Im} N_J^{\mathbb{C}})\\ &\le n - \tfrac{1}{2}\operatorname{rank}\,(N_J)_x \qquad (\dagger\dagger)\end{aligned}$$

The conclusion follows directly by combining $(\dagger\dagger)$ and $(\dagger)$. $\square$

We now give an example to show that the function $x \mapsto \operatorname{rank}(\operatorname{Im} N_J)_x$ does not control the number of J-holomorphic functions near a point, in general. A related question for which we do not have an answer is:

Question 1. Given an almost complex manifold (M, J), does the function $x \mapsto \operatorname{rank}(\mathcal{D}_J^\infty)_x$ control the number of functionally independent J-holomorphic functions?

The following lemma is useful when constructing (local) almost complex structures:

Lemma 4. Let J be an almost complex structure on $\mathbb{R}^{2n}$ such that $J(0) = J_0$. A $C^\infty(M,\mathbb{C})$-basis for the space of $(0,1)$-vector fields on an open neighbourhood U of 0 in $\mathbb{R}^{2n}$ is given by

$$\partial_{\bar z_j} + \sum_{i=1}^n a_j^i\,\partial_{z_i} \quad j = 1,\dots,n \qquad (3)$$

where the a_i^j are smooth functions satisfying $a_i^j(0) = 0$. A basis for the $(1,0)$-vector fields is obtained by conjugation.

$$\partial_{z_j} + \sum_{i=1}^n \bar a_j^i\,\partial_{\bar z_i} \quad j = 1,\dots,n. \qquad (4)$$

Conversely, a set of vector fields of the form (3) determines a unique almost complex structure J with (0,1)-vector fields given by (3) such that $J(0) = J_0$.

This lemma is well known (see [20], eqn. 1.3.), but we were unable to locate a proof in the literature, here is a quick way to see it.

Proof. Let $T\mathbb{R}^{2n}_{\mathbb{C}} = T\mathbb{R}^{1,0} \oplus T\mathbb{R}^{0,1}$ be the splitting with respect to the *standard* complex structure J_0, and $T\mathbb{R}^{2n}_{\mathbb{C}} = E^{1,0}\oplus E^{0,1}$ the splitting with respect to J. Since $J(0) = J_0$, $E_x^{0,1}$ and $T\mathbb{R}_x^{1,0}$ are transversal subspaces of $(T\mathbb{R}^{2n}_{\mathbb{C}})_x$, where x belongs to a small enough neighbourhood U of 0. Hence, there is a unique linear endomorphism $A_x : T\mathbb{R}_x^{0,1} \to T\mathbb{R}_x^{1,0}$ whose graph is $E_x^{0,1}$. A_x varies smoothly in x because J is smooth. Then $A(\partial_{\bar z_j}) = \sum_{i=1}^n a_j^i \partial z_i$ for some smooth $a_i^j : U \to \mathbb{C}$. Note that $a_i^j(0) = 0$ since $A_0(\partial_{\bar{z}_i}) = 0$. The a_i^j are thus the coefficient of a basis of $\Gamma(E^{0,1})$, as in (3).

To see that a basis of the form (3) determines a unique almost complex structure J on U, denote by $B : T\mathbb{R}^{0,1} \to T\mathbb{R}^{2n}_{\mathbb{C}}$ the linear map determined by $\partial_{\bar{z}_i} \mapsto \partial_{\bar{z}_j} + \sum_{i=1}^n a_j^i \partial_{z_i}$ and $\bar{B} : T\mathbb{R}^{1,0} \to T\mathbb{R}^{2n}_{\mathbb{C}}$ the conjugate map defined in the obvious way. Notice that $(B, \bar{B}) : T\mathbb{R}^{2n}_{\mathbb{C}} \to T\mathbb{R}^{2n}_{\mathbb{C}}$ is invertible, since the images of B and $\bar{B}$ are transversal subspaces of $T\mathbb{R}^{2n}_{\mathbb{C}}$. If an almost complex structure J with $-i-$eigenspace given by (3) exists, it must satisfy the equation

$$J(B, \bar{B}) = (-iB, i\bar{B}),$$

hence $J = (-iB, i\bar{B})(B, \bar{B})^{-1}$. (Note that J thus defined is an endomorphism of $T\mathbb{R}^{2n}_{\mathbb{C}}$, but it is also the complexification of the almost complex structure j of $T\mathbb{R}^{2n}$ defined by $j\ \mathrm{Re}(B) := \mathrm{Im}(B)$ and $j\ \mathrm{Im}(B) := -\mathrm{Re}(B)$.) $\square$

Remark 4. From the proof of the second part of the lemma, it is easily seen that any set of complex vector fields $X_1, \ldots, X_n$ such that $X_1, \ldots, X_n, \bar{X}_1, \ldots, \bar{X}_n$ is linearly independent also determines an almost complex structure with $(0,1)$-vector fields given by $\bar{X}_1, \ldots, \bar{X}_n$.

Observe that given a basis $X_1^{0,1}, ..., X_n^{0,1}$ of $TM^{0,1}|_U$, a smooth function $f : U \to \mathbb{C}$ is $J-$holomorphic if and only if $X_i^{0,1}(f) \equiv 0$ for $i = 1, \ldots, n$. (Note that this equation means $df(X_i^{0,1}) = 0$, where $df_p : T_pM_{\mathbb{C}} \to \mathbb{C}$ is obtained by extending the usual exterior derivative $\mathbb{C}-$linearly). Thus, the vector fields of (3) in lemma 4 should be seen as a system of first-order PDOs characterizing J-holomorphic functions.

The example is based on the following theorem of L. Nirenberg.

Theorem 2 (Nirenberg, [22], Theorem 1)**.** Let $z = x + iy = re^{i\theta}$ and t be coordinates on $\mathbb{R}^3$. There exists smooth functions $\phi, \psi : \mathbb{R}^3 \to \mathbb{R}$ such that the operator

$$P = \partial_{\bar{z}} + iz\, \partial_t + z\varphi(z,t)\, \partial_t + z\psi(z,t)\, \partial_\theta \tag{5}$$

is such that if u is a smooth solution to $Pu = 0$, then u is constant.

Corollary 1. Let $z_1 = x_1 + iy_1 = r_1e^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2e^{i\theta_2}$ and φ, ψ as in Nirenberg's theorem above. The vector fields

$$X_1 = \partial_{\bar{z}_1} + iz_1\partial_{x_2} + z_1\varphi(z_1, x_2)\partial_{x_2} + z_1\psi(z_1, x_2)\partial_{\theta_1}$$
$$X_2 = \partial_{\bar{z}_2} + iz_2\partial_{x_1} + z_2\varphi(z_2, x_1)\partial_{x_1} + z_2\psi(z_2, x_1)\partial_{\theta_2}$$

define a basis for the $(0,1)$-vectorfield of an almost complex structure J in a neighbourhood of the origin in $\mathbb{R}^4$. The Nijenhuis tensor satisfies $\mathrm{rank}_{\mathbb{R}}\,(\mathrm{Im}\, N_J)_x \leq 2$ everywhere and there exists no J-holomorphic functions with domain a neighbourhood of the origin.

Proof. By remark 4, X_1 and X_2 induces an almost complex structure J on a neighbourhood of the origin if $X_1, X_2, \bar{X}_1$ and $\bar{X}_2$ are linearly independent at the origin. Note that at the origin, X_1, X_2 equals $\partial_{\bar{z}_1}, \partial_{\bar{z}_2}$, so linear independence is obvious. The fact that $\mathrm{rank}_{\mathbb{R}}\, N_J \leq 2$ is a general property of almost complex structure in dimension four. Indeed, N_J can be seen as a complex linear map $T\mathbb{R}^{1,0}_J \wedge T\mathbb{R}^{1,0}_J \to T\mathbb{R}^{0,1}_J$, the domain of this map has complex dimension one, so the conclusion follows. Finally, suppose f is a J-holomorphic function, then $X_1 f = X_2 f = 0$. Since $X_1|_{\{y_2=\mathrm{cst}\}}$ belongs to the tangent space of any hypersurface $\{y_2 = \mathrm{cst}\}$, it is also true that $X_1|_{\{y_2=\mathrm{cst}\}}(f|_{\{y_2=\mathrm{cst}\}}) = 0$. Theorem 2 above then implies that $f(x_1, y_1, x_2, \mathrm{cst})$ is a constant function. Similarly, we get $f(x_1, \mathrm{cst}, x_2, y_2)$ is a constant function. Then $f(0) = f(x_1, 0, x_2, y_2) = f(x_1, y_1, x_2, y_2)$ for any point (x_1, y_1, x_2, y_2) on which f is defined. $\square$

Now we have :

Proposition 2. Let $\hat{J}$ be the almost complex structure of the previous corollary. There exists an almost complex structure J which possesses non-constant J-holomorphic functions in a neighbourhood of the origin and such that rank $(N_J)_x = \text{rank } (N_{\hat{J}})_x$ for all x in that neighbourhood.

The following lemma will be convenient to prove the proposition:

Lemma 5. Let F be any compact subset of $\mathbb{C}^2$, there exists an almost complex structure J defined on some open neighbourhood of F such that $\mathrm{rank}_{\mathbb{R}}\, (N_J)_x = 0$ on F and 1 elsewhere. Moreover J admits non-constant J-holomorphic functions in a neighbourhood of any points.

Proof. Since F is compact, it is contained in some polydisk $D(0, R) = D(0, R_1) \times D(0, R_2)$ of radius $R > 0$. Let $\rho : \mathbb{C}^2 \to \mathbb{R}$ be a smooth function vanishing on F and greater than 0 on the complement of F.[4] Let $\mu(z_1, z_2)$ be a solution to the non-homogeneous problem

$$\begin{cases} \partial_{\bar{z}_1} \mu(z_1, z_2) = \rho(z_1, z_2). \\ \mu(0) = 0 \end{cases}$$

in $D(0, R)$. (Solutions to this problem are not hard to come by, for example $\mu(z_1, z_2) := \frac{1}{2i} \int_{D_{(0,R_1)}} \frac{\rho(\zeta, z_2)}{\zeta - z_1} d\zeta d\bar{\zeta}$ is one up to constant, see [11], Prop. 1.3.7.) Now considere the vector fields

$$\begin{aligned} X_1 &= \partial_{\bar{z}_1} \\ X_2 &= \partial_{\bar{z}_2} + \mu(z_1, z_2) \partial_{z_1} \end{aligned} \tag{6}$$

in $D(0, R)$. By lemma 4, X_1 and X_2 defines a basis for the $(0, 1)$-vector fields of some almost complex structure J on $D(0, R)$. It is clear that $\mathrm{rank}_{\mathbb{C}}\, (N_J^{\mathbb{C}})_x \leq 1$ with equality if and only if $N_J^{\mathbb{C}}(\bar{X}_1, \bar{X}_2)_x = [\bar{X}_1, \bar{X}_2]^{0,1} \neq 0$. We then compute

$$[\bar{X}_1, \bar{X}_2]^{0,1} = [\partial_{z_1}, \partial_{z_2} + \bar{\mu}\, \partial_{\bar{z}_1}] = (\partial_{z_1} \bar{\mu})\, \partial_{\bar{z}_1} = \bar{\rho}\, \partial_{\bar{z}_1} = \rho\, \partial_{\bar{z}_1},$$

which vanishes exactly on F. Finally, note that any holomorphic function $f(z_2) : D(0, R_2) \to \mathbb{C}$ is a solution to (6). $\square$

proof of proposition 2. Notice that the locus $F = \{x \in \mathbb{R}^4 \mid \text{rank } (N_{\hat{J}})_x = 0\}$ is closed, since $N_{\hat{J}}$ is a smooth tensor. Let D be a closed bounded disk centered at the origin. Since $D \cap F$ is compact, the lemma implies that there exists an almost complex structure J defined on a neighbourhood of $F \cap D$ for which rank $N_{\hat{J}} = \mathrm{rank} N_J$ on $D \cap F$. $\square$

[4]If existence is unclear, see [16] thm 2.29.

The goal of this section is to state and prove some properties of the J-holomorphic functions that will be usefull in part II. We first describe a close relation between the fibers of a J-holomorphic functions and $\mathcal{D}_J^\infty$.

Lemma 6 (cf. Muskarov, [19], lemma 2.3)**.** Let (M, J) be an almost complex manifold. The association $V \mapsto V_\mathbb{R}$ is a bijection between involutive complex distributions of $TM_\mathbb{C}$ containing $TM^{1,0}$ and real distributions F of TM satisfying the following properties:

i. F is J-invariant, i.e. $JF \subset F$.
ii. $\operatorname{Im} N_J \subset F$.
iii. F is involutive.
iv. $(\mathcal{L}_X J)Y \in \Gamma(F)$ for all $X \in \Gamma(F)$ and all $Y \in \Gamma(TM)$.

Proposition 3 (compare with CGG [4] Proposition 1.3.)**.** Let (M, J) be an almost complex manifold and let $\mathcal{D}_J^{(1)} := \operatorname{Im} N_J$. Define inductively

$$\mathcal{D}_J^{(k+1)} := \mathcal{D}_J^{(k)} + \sum_{X \in \mathcal{D}_J^{(k)}} \operatorname{Im}(\mathcal{L}_X J) + [\mathcal{D}_J^{(k)}, \mathcal{D}_J^{(k)}].$$

Then $\mathcal{D}_J^\infty = \bigcup_{k=1}^\infty \mathcal{D}_J^{(k)}$.

Proof. We will show that $\mathcal{D}_J^\infty$ and $\cup_{k=1}^\infty \mathcal{D}_J^{(k)}$ are both the smallest real distribution in TM satisfying properties i.-iv. of Lemma 6. First, it is clear that the correspondence $V \mapsto V_\mathbb{R}$ of Lemma 6 is inclusion preserving. Since $(TM^{1,0})^\infty$ is the smallest involutive distribution containing $TM^{1,0}$, $D_J^\infty = ((TM^{1,0})^\infty)_\mathbb{R}$ is the smallest involutive distribution satisfying properties i.-iv. It is shown in [4] Prop. 1.3-1.4 that $\cup_{k=1}^\infty \mathcal{D}_J^{(k)}$ satisfies i.-iv., the only point missing in [4] is that $\cup_{k=1}^\infty \mathcal{D}_J^{(k)}$ is the smallest such distribution. To prove this, let T be any distribution satisfying i.-iv., then $\mathcal{D}_J^{(1)} = \operatorname{Im} N_J \subset T$ by ii. Suppose for induction that $\mathcal{D}_J^{(k)} \subset T$. Then since T is involutive (property iii.), $[\mathcal{D}_J^\infty, \mathcal{D}_J^\infty] \subset [T, T] \subset T$. Also for each $X \in \mathcal{D}_J^{(k)}$, $\operatorname{Im} \mathcal{L}_X J \subset T$ by property iv. Hence

$$\mathcal{D}_J^{(k+1)} = \mathcal{D}_J^{(k)} + \sum_{X \in \mathcal{D}_J^{(k)}} \operatorname{Im}(\mathcal{L}_X J) + [\mathcal{D}_J^{(k)}, \mathcal{D}_J^{(k)}] \subset T.$$

□

Theorem 3. Let (M, J) be an almost complex manifold and let $f \in \mathcal{O}_M^J(U)$ be a J-holomorphic function. The function f is constant along $\mathcal{D}_J^\infty$, i.e. for any $X \in \mathcal{D}_J^\infty(U)$, one has $X(f) = 0$.

Proof. We proceed by induction. We first show that $X(f) = 0$ for each $X \in \mathcal{D}_J^{(1)} = \operatorname{Im} N_J$. Recall that $JX(f) = iX(f)$ for any vector field X by $J-$holomorphicity of f. Let $X, Y \in \Gamma(TM, U)$,

$$\begin{aligned} N_J(X,Y)f &= [X,Y]f + J[JX,Y]f + J[X,JY]f - [JX,JY]f \\ &= (XYf - YXf) + (iJXYf + YXf) \\ &\quad + (-XYf - iJYXf) - (iJXYf - iJYXf) \end{aligned}$$

Each term in the last sum cancels out, so $N_J(X,Y)f = 0$. Suppose for induction that for all $X \in \Gamma(\mathcal{D}_J^{(k)}, U)$, $X(f) = 0$. Let $[X,Y] \in \Gamma([\mathcal{D}_J^{(k)}, \mathcal{D}_J^{(k)}], U)$, then $[X,Y]f = XYf - YXf = 0$. Let $X \in \Gamma(\mathcal{D}_J^{(k)}, U)$ and $Y \in \Gamma(TM, U)$, then

$$\begin{aligned} ((\mathcal{L}_X J)Y)f &= [X, JY]f - J[X,Y]f \\ &= iXYf - JYXf - iXYf + iYXf \\ &= (iY - JY)Xf \\ &= 0 \quad \text{because} \quad X(f) = 0. \end{aligned}$$

It follows that f vanishes on $\mathcal{D}_J^{(k+1)}$ by Proposition 3, which completes the induction step. $\square$

We then have the following local structure theorem when (M, J) is of constant type

Theorem 4. Let (M, J) be an almost complex manifold and suppose that J is of type m at a point $p \in M$. Then there exist open subsets $p \in U \subset M$ and $0 \in V \subset \mathbb{C}^m$ together with a (J, i)-holomorphic surjective submersion $\pi : U \to V$ that induces isomorphisms of sheaves

$$\pi^* : \mathcal{O}_{\mathbb{C}^m}|_V \to \pi_* \mathcal{O}_M^J|_U \quad \text{and} \quad \pi^* : \pi^{-1}\mathcal{O}_{\mathbb{C}^m}|_V \to \mathcal{O}_M^J|_U. \tag{7}$$

Moreover, $\ker d\pi_p = (\mathcal{D}_J^\infty)_p$ for each $p \in U$.

Corollary 2. If (M, J) is an almost complex manifold and J is of type m at $p \in M$, then $\mathcal{O}_{M,p}^J \cong \mathcal{O}_{\mathbb{C}^m,0}$.

Proof. Since $\pi^* : \pi^{-1}\mathcal{O}_{\mathbb{C}^m}|_V \to \mathcal{O}_M^J$ is an isomorphism, it induces isomorphisms on stalks, which gives the conclusion since $(\pi^{-1}\mathcal{O}_{\mathbb{C}^m})_p = \mathcal{O}_{\mathbb{C}^m, \pi(p)}$. $\square$

Remark 5. the second isomorphism in (7) implies that π is a flat[5] morphism of locally ringed space.

proof of theorem 4. Since J is of type m at p, there exist exactly m linearly independent J-holomorphic functions in a neighbourhood of p. Let $f^1, ..., f^m$ be such functions. By adding constants if necessary, we can assume that $f^i(p) = 0$ for each $i = 1, \dots, m$. Note that the maps f^i remain linearly independent in a neighbourhood U of p. These functions define a submersion $\pi : U \to \mathbb{C}^m$ where $\pi = (f^1, \dots, f^m)$. Since π is a submersion, its image, denoted V, is open. If necessary, shrink the domain U of π so that all the fibres of π are connected (this is possible because of the local form of a submersion). Since $\pi : U \to V$ is surjective, it induces an injective morphism of sheaves of ring $\pi^* : \mathcal{O}_{\mathbb{C}^m}|_V \to \pi_* \mathcal{O}_M^J|_U$. To prove that this morphism is surjective, it suffices to check that the map is surjective at the level of stalks. This translate into the following lifting problem: for every $p \in M$ and every section $\varphi \in \mathcal{O}_M^J(\pi^{-1}(W))$ where $p \in W \subset V$ is open, we must show there exists a perhaps smaller open subset $p \in W' \subset W$ and a unique holomorphic map $\psi : W' \to \mathbb{C}$ such that:

$$\begin{tikzcd} (\pi^{-1}(W'), J) \arrow[r, "\pi"] \arrow[d, "\varphi"'] & (W', i) \arrow[dl, dashed, "\exists!\,\psi"] \\ (\mathbb{C}, i) & \end{tikzcd} \tag{$\star$}$$

[5]see [27] for the definition and basic properties of flatness. This is used in Part 2, §4.6.

commutes. We will instead solve a slightly more general lifting problem. For every $p \in M$ and every section $\varphi \in \mathcal{O}_M^J(U')$ where $p \in U' \subset U$ is open and $\pi^{-1}(x) \cap U'$ is connected for each $x \in \pi(U')$, we show that there exists a holomorphic map ψ such that the diagram

$$\begin{tikzcd} (U', J) \arrow[r, "\pi"] \arrow[d, "\varphi"'] & (\pi(U'), i) \arrow[dl, dashed, "\exists!\, \psi"] \\ (\mathbb{C}, i) & \end{tikzcd} \tag{$\star\star$}$$

commutes. (Solving $(\star\star)$ implies solving $(\star)$: given $\varphi \in \mathcal{O}_M^J(\pi^{-1}(W))$, it suffices to let $W' := W$ and $U' := \pi^{-1}(W)$)

Let $z \in \pi(U')$, we claim that $\varphi|_{\pi^{-1}(z)}$ is constant. Since z is a regular value of π and $\pi^{-1}(z) \cap U'$ is connected, it suffices to show that $T_p\pi^{-1}(z) = \ker(d\pi_p)$ is a subset of $\ker(d\varphi_p)$ for each $p \in \pi^{-1}(z)$. Observe that since φ is J-holomorphic, it is functionally dependent on $f^1, ..., f^m$ for each $p \in U$. More precisely, there are smooth functions $a_1, ..., a_n \in C^\infty(U)$ such that $d\varphi_p = \sum_i a_i(p) df_p^i$. It follows that $\ker(d\pi_p) = \cap_{i=1}^n \ker(df_p^i) \subset \ker(d\varphi_p)$. The map $\psi : W \to \mathbb{C}$ defined by $\psi(z) = \varphi(p)$ where $p \in \pi^{-1}(z)$ is then well defined.

To see that ψ is smooth near any $z \in W$, let s be a smooth local section[6] of π defined on a neighbourhood of z. From the relation $\psi \circ \pi = \varphi$, we get $\psi = \varphi \circ s$ on a neighbourhood of z. The map $\varphi \circ s$ is smooth so ψ is smooth. Finally, to check that ψ is holomorphic, let $M \in T_pW$ and $Y \in T_qU$, where $\pi(q) = p$ and $d\pi_q Y = X$. We have

$$\begin{aligned} d\psi_p(iX) &= d\psi_p df_q(JY) \quad \text{by } J\text{-holomorphicity of } \pi \\ &= id\psi_p df_q(Y) \quad \text{by J-holomorphicity of } \varphi \\ &= id\psi_p(X) \end{aligned}$$

which yields the holomorphicity of ψ. We conclude that $\pi^* : \mathcal{O}_{\mathbb{C}^m}|_V \to \pi_*\mathcal{O}_M^J|_U$ is an isomorphism.

We are left to show that $\pi^* : \pi^{-1}\mathcal{O}_{\mathbb{C}^m}^J \to \mathcal{O}_M^J$ is an isomorphism. Since π is open, note that $\pi^{-1}\mathcal{O}_{\mathbb{C}^m}^J(V') = \mathcal{O}_{\mathbb{C}^m}^J(\pi(V'))$ and $\pi^*(\varphi) = \varphi \circ \pi$ for any $\varphi \in \pi^{-1}\mathcal{O}_{\mathbb{C}^m}(V')$. Injectivity of π^* is obvious, and surjectivity follows directly from the lifting problem $(\star\star)$ which we solved. Finally, to show that $\ker d\pi_p = (\mathcal{D}_J^\infty)_p$, observe that theorem 3 implies that $\ker d\pi_p \subset (\mathcal{D}_J^\infty)_p$. Since J has constant type, $(TM^{1,0})^\infty$ is regular and thus corollary 1 implies that $\mathcal{D}_J^\infty$ is also regular with $\mathrm{rank}(\mathcal{D}_J^\infty)_x = 2(n-m)$, but also $\mathrm{rank}(\ker d\pi_x) = 2(n-m)$ and we conclude $\mathrm{rank}(\ker d\pi_x) = (\mathcal{D}_J^\infty)_x$. □

To end the section, we remark the following property of J-holomorphic functions:

Lemma 7 (Unique continuation)**.** Let $f \in \mathcal{O}_M^J(U)$ where $U \subset M$ is a connected open subset and suppose that f vanishes to infinite order at a point $p \in U$. Then $f \equiv 0$ on U.

[6]This is a smooth map s defined in a neighbourhood of z such that $\pi \circ s = \mathrm{Id}$. Existence is proven using the standard local form of a submersion.

Since we will not be using this result, we defer the proof to Appendix I. It follows that $\mathcal{O}_M^J$ is an integral sheaf of rings and thus, there is a well defined notion of sheaf of meromorphic functions on any almost complex manifold.

Part II

We now tackle global questions related to J-holomorphic functions. The first step is to define the global objects we shall study. We with the definition of Basic Dolbeault cohomology.

4 BASIC DOLBEAULT COHOMOLOGIES

There are two closely related Dolbeault-type cohomology theories that are concerned with complex structures on distributions in the literature. The first was introduced by E.K. Alaoui in 1990 [13], and three decades later, a second theory was proposed by Cahen–Gutt–Gutt [4].

The two theories are built following the same steps: one starts with a manifold equipped with an involutive distribution $(M, \mathcal{D})$ and defines the data of a "complex structure transverse to $\mathcal{D}$". Such complex structure is designed so that $\bar\partial\bar\partial = 0$ on complex valued $\mathcal{D}$-basic forms. The Dolbeault-type cohomology is then obtained by taking the cohomology of $\bar\partial$ on the cdga of $\mathcal{D}$-basic forms.

The two theories do not generally agree. The fundamental differences are condensed in the following table:

	E.K. Alaoui	Cahen-Gutt-Gutt
Hypotheses on $\mathcal{D}$	Regular, Riemannian	None
The complex transverse structure is an endomorphism of	$TM/\mathcal{D}$	TM

The theory of E.K. Alaoui is easily generalized so that no hypotheses is assumed on $\mathcal{D}$. Then CGG's construction becomes a special case of E.K. Alaoui's. Since E.K. Alaoui's work is not mentioned in CGG's more recent work, we explain how the two theories are related in section 4.1.

4.1 **Basic Dolbeault Cohomology**

Let $\mathcal{D}$ be an involutive distribution on a smooth manifold M. If $\mathcal{D}$ is regular then $Q := TM/\mathcal{D}$ is a smooth vector bundle. Notice that because $\mathcal{D}$ is involutive, the Lie brackets $[\_,\_]$ of $\Gamma(TM)$ descend to a $\mathbb{R}$-bilinear map $[\_,\_] : \Gamma(D)\times\Gamma(Q) \to \Gamma(Q)$. E.K. Alaoui defines a *transverse almost complex structure* to be a bundle morphism $J : Q \to Q$ which is holonomy invariant, i.e., $\mathcal{L}_X J(Y) := [X, JY] - J[X, Y] = 0$ for all $X \in \mathcal{D}$, $Y \in \Gamma(Q)$. The Lie brackets $[\_,\_]$ of $\Gamma(TM)$ *do not* descend to a $\mathbb{R}$-bilinear map $[\_,\_] : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$, but the Nijenhuis tensor N_J does, by holonomy invariance of J. E.K. Alaoui defines a *transverse complex structure* to be a transverse almost complex structure for which N_J vanishes.

We now generalize these definitions to non-regular $\mathcal{D}$. The issue to overcome is that $TM/\mathcal{D}$ is not a smooth vector bundle (it may have non-constant rank). The classical way to handle this is to enlarge the category to that of sheaves of C^∞_M-modules. We thus let $\mathcal{T}M$ be the sheaf of sections of TM

and identify $\mathcal{D}$ with its sheaf of sections. We let $\mathcal{Q}$ be the quotient sheaf $\mathcal{T}M/\mathcal{D}$. Note that since we are assuming M is paracompact, and that $\mathcal{T}M$ and $\mathcal{D}$ are sheaves of C^∞_M-module, the presheaf $U \mapsto \Gamma(U,\mathcal{T}M)/\Gamma(U,\mathcal{D})$ is a sheaf (without requiring sheafification) and is canonically isomorphic to $\mathcal{Q}$. The Lie brackets $[\_,\_]$ of $\mathcal{T}M$ are regarded as a morphism of sheave $\mathcal{T}M\times\mathcal{T}M \to \mathcal{T}M$ and descend to a $\mathbb{R}$-bilinear morphism $[\_,\_] : \mathcal{D}\times\mathcal{Q} \to \mathcal{Q}$. Then we define:

Definition 4. A *transverse almost complex structure* on $(M,\mathcal{D})$ is a morphism of $C^\infty(M,\mathbb{R})$-modules $J : \mathcal{Q} \to \mathcal{Q}$ such that for all open sets U of M, $J_U \circ J_U = -\mathrm{Id}_U$. We also require that J be holonomy invariant, meaning that the morphism $[\_, J\_] - J[\_,\_] : \mathcal{D} \times \mathcal{Q} \to \mathcal{Q}$ is equal to the zero morphism.

Given a transverse almost complex structure J, we can define the Nijenhuis tensor $N_J(\_,\_) : \mathcal{Q}\times\mathcal{Q} \to \mathcal{Q}$ in the evident way. It is again well defined by holonomy invariance of J.

Definition 5. A *transverse complex structure* J on $(M,\mathcal{D})$ is a transverse almost complex structure for which N_J is equal to the zero morphism.

Remark 6. Whenever $\mathcal{D}$ is regular, our definition of a transverse complex structure agrees with the one given by E.K. Alaoui.

Remark 7. CGG defines a transverse complex structure as follows: Denote by $\Gamma(\mathcal{D})$ the global sections of $\mathcal{D}$. A transverse complex structure is an endomorphism J of TM such that (i) $J(\Gamma(\mathcal{D})) \subset \Gamma(\mathcal{D})$, (ii) $J^2Y + Y \in \Gamma(\mathcal{D})$, (iii) $[X, JY] - J[X,Y] \in \Gamma(\mathcal{D})$ for all $X \in \Gamma(\mathcal{D})$ and all $Y \in \Gamma(TM)$, and (iv) $N_J(U,V) \in \Gamma(\mathcal{D})$ for all $U, V \in \Gamma(TM)$. Since $\mathcal{D}$ and $\mathcal{T}M$ are fine sheaves over paracompact spaces, conditions (i-iv) automatically holds for sections over arbitrary open subset of M. Then conditions (i-iii) imply that J descends to a transverse almost complex structure $J : \mathcal{Q} \to \mathcal{Q}$ (in the sense of Definition 4) and condition (iv) implies that $J : \mathcal{Q} \to \mathcal{Q}$ is a transverse complex structure.

We now define Basic Dolbeault cohomology. The construction is very similar to the usual Dolbeault cohomology. We only have to be careful that everything carries over to possibly singular distributions. Let $\mathcal{Q}_\mathbb{C} := \mathcal{Q}\otimes\mathbb{C}$. This fits canonically in the exact sequence

$$0 \to \mathcal{D}\otimes\mathbb{C} \to \mathcal{T}M\otimes\mathbb{C} \to \mathcal{Q}_\mathbb{C} \to 0.$$

Lemma 8. Let $(M,\mathcal{D})$ be a smooth manifold equipped with an involutive distribution and J a transverse almost complex structure. There is a direct sum decomposition $\mathcal{Q}_\mathbb{C} \cong \mathcal{Q}^{1,0} \oplus \mathcal{Q}^{0,1}$ of $C^\infty_{M,\mathbb{C}}-$modules such that the $\mathbb{C}$-linear extension of J acts as multiplication by i (resp. $-i$) on $\mathcal{Q}^{1,0}$ (resp. $\mathcal{Q}^{0,1}$).

Proof. Observe that $\mathcal{Q}^{1,0}(U) := \{X - iJX \mid X \in \mathcal{Q}(U)\} \subset \mathcal{Q}_\mathbb{C}(U)$ (and $\mathcal{Q}^{0,1}$ defined analogously) is a sheaf of $C^\infty_{M,\mathbb{C}}$-modules and gives the desired direct-sum decomposition. □

Definition 6. For a transverse almost complex structure J on $(M,\mathcal{D})$, we define the sheaves of $C^\infty_{M,\mathbb{C}}$-modules

$$\mathcal{A}^n_Q := \bigwedge^n (Q_\mathbb{C})^* \quad \text{and} \quad \mathcal{A}^{r,s}_Q := \bigwedge^r (Q^{1,0})^* \otimes \bigwedge^s (Q^{0,1})^*$$

Lemma 9. There is a direct sum decomposition of $C^\infty_{M,\mathbb{C}}-$modules

$$\mathcal{A}^n_Q \overset{\sim}{\to} \bigoplus_{r+s=n} \mathcal{A}^{r,s}_Q$$

It follows that there are canonical morphisms of $C^\infty_{M,\mathbb{C}}$-modules given by projections : $\Pi^{r,s} : \mathcal{A}^{r+s}_Q \to \mathcal{A}^{r,s}_Q$.

Proof. We have to be slightly careful since such decomposition is generally proven over vector bundles via an argument relying on the finite dimensional vector space structure of the fibres. First there is a bilinear morphism of sheaves

$$\underbrace{(\mathcal{Q}^{1,0})^*\oplus(\mathcal{Q}^{0,1})^* \times \cdots \times (\mathcal{Q}^{1,0})^*\oplus(\mathcal{Q}^{0,1})^*}_{\text{n times}} \to \bigoplus_{n=r+s} \mathcal{A}^{r,s}_{\mathcal{Q}}$$

given on an open set U by mapping $(x_1 + y_1, ..., x_n + y_n)$ to $\sum_{I+J=\{1,...,n\}}(-1)^{\sigma(I,J)}x_I \wedge y_J$ where $x_I = x_{i_1} \wedge \cdots \wedge x_{i_{|I|}}$ (similarly for y_J) and $\sigma(I, J)$ is the number of transpositions needed to order the x_{i_j} and y_{i_j} in a sequence such that the indices i_j are in increasing order (e.g. $y_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge y_n$). The universal property of tensor product then gives a morphism $\mathcal{A}^n_{\mathcal{Q}} \overset{\sim}{\to} \bigoplus_{r+s=n} \mathcal{A}^{r,s}_{\mathcal{Q}}$. We can check that this is an isomorphism at the level of stalks, which reduces the problem to the statement that if M and N are k-modules where k is a commutative ring, then the map $(x+y) \to \sum_{I+J=\{1,...,n\}}(-1)^{\sigma(I,J)}x_I \wedge y_J$ induces an isomorphism of modules $\Lambda^p(M\oplus N) \to \bigoplus_{m+n=p} \left(\Lambda^m M\otimes\Lambda^n N\right)$. This is well known. (see [3] Theorem 7) $\square$

Definition 7. Let $(M, \mathcal{D})$ be a smooth manifold with an involutive distribution $\mathcal{D}$. Given $\omega \in \Gamma(\mathcal{A}^k_{\mathcal{Q}}, U)$, $Y \in \Gamma(\mathcal{D}_{\mathbb{C}}, U)$ and $X_1, ..., X_k \in \Gamma(\mathcal{Q}, U)$, we let

$$\mathcal{L}_Y\omega := Y(\omega(X_1, ..., X_k)) - \textstyle\sum_{i=1}^k \omega(X_1, \dots, [Y, X_i], \dots, X_n)$$

Where $[\_, \_]$ is the bracket $[\_, \_] : \mathcal{D}_{\mathbb{C}} \times \mathcal{Q}_{\mathbb{C}} \to \mathcal{Q}_{\mathbb{C}}$.

Definition 8. Let $(M, \mathcal{D})$ be a smooth manifold with an involutive distribution. The sheaf of basic p-forms $\mathcal{A}^p_B$ is given by

$$\mathcal{A}^p_B(U) := \{\omega \in \mathcal{A}^p_Q(U) \ : \ \mathcal{L}_X\omega = 0 \text{ for all } X \in \mathcal{D}(U)\}.$$

If J is a transverse almost complex structure on $(M, \mathcal{D})$, then the sheaf of basic $(r, s)-$forms is given by $\mathcal{A}^{r,s}_B := \mathcal{A}^{r,s}_Q \cap \mathcal{A}^{r+s}_B$.

Definition 9. We define the exterior derivative d on $\mathcal{A}^p_B$ to be the morphism of sheaves (of abelian groups) $d : \mathcal{A}^k_B \to \mathcal{A}^{k+1}_B$ given by the invariant formula: if $\omega \in \mathcal{A}^k_B(U)$ and $X_1, \dots, X_{k+1} \in \mathcal{Q}(U)$, then

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) := &\sum_{1\le i\le k+1} (-1)^{i-1} X_i(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})) \\ &+ \sum_{1\le i<j\le k+1} (-1)^{i+j}\omega([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}), \end{aligned} \tag{8}$$

We then let $\partial := \Pi^{r+1,s} \circ d$ and $\bar\partial := \Pi^{r,s+1} \circ d$.

Note that d is well defined on $\mathcal{A}_B^\bullet$ by holonomy invariance of the basic forms (altough one cannot define d on $\mathcal{A}_Q$ using equation (8)).

Lemma 10. If ω is a basic k-form (resp. (r,s)-form) then $d\omega$ is a basic $k{+}1$-form (resp $\bar{\partial}\omega$ is a $(r,s{+}1)$-form). Thus the morphisms $d : \mathcal{A}_B^k \to \mathcal{A}_B^{k+1}$ and $\bar{\partial} : \mathcal{A}_B^{r,s} \to \mathcal{A}_B^{r,s+1}$ are well defined.

Proof. First, note that Cartan's formula $\mathcal{L}_Y\omega = \iota_Y d\omega + d\iota_Y\omega$ holds. It follows from a symbolic computation using definition of $\mathcal{L}$ and d. Then if $\mathcal{L}_X\omega = 0$, we have $d\iota_X\omega = -\iota_X d\omega$ and $\mathcal{L}_X d\omega = d\iota_X d\omega = -\iota_X dd\omega = 0$. To show that $\bar{\partial}\omega$ is basic whenever $\omega \in \Gamma(\mathcal{A}_B^{r,s}, U)$, observe that $0 = \mathcal{L}_X d\omega = \mathcal{L}_X\bar{\partial}\omega + \mathcal{L}_X\partial\omega$ so it suffices to show that $\mathcal{L}_X\bar{\partial}\omega$ (resp. $\mathcal{L}_X\partial\omega$) is a $(r,s{+}1)$-form (resp. a $(r{+}1,s)$-form) for each $X \in \Gamma(\mathcal{D}, U)$. Start with $\mathcal{L}_X\bar{\partial}\omega$. we must check that $\mathcal{L}_X\omega(X_1, ..., X_{p+q+1}) = 0$ where the X_i's are either $(1,0)$ or $(0,1)$ vector and the number of $(1,0)$-vector is not equal to p. Assume $X_1, ..., X_k$ are of type $(1,0)$ where $k \neq p$. Then

$$\mathcal{L}_X\bar{\partial}\omega(X_1, .., X_{p+q+1}) = X(\bar{\partial}\omega(X_1, .., X_{p+q+1})) + \sum_{i=1}^{k} \bar{\partial}\omega(X_1, .., [X, X_i], .., X_{p+q+1})$$

we claim that $[X, X_i]$ is of type $(1,0)$ (resp. $(0,1)$) if X_i is of type $(1,0)$ (resp. $(0,1)$). Indeed, since J is holonomy invariant, we have $0 = (\mathcal{L}_X J)X_i = [X, JX_i] - J[X, X_i]$, so that if X_i is of type $(1,0)$, then $J[X, X_i] = [X, JX_i] = i[X, X_i]$ (and similarly for type $(0,1)$). It immediatly follows that $\mathcal{L}_X\bar{\partial}\omega(X_1, ..., X_{p+q+1}) = 0$, yielding the conclusion. (The proof that $\mathcal{L}_X\partial\omega$ is a $(r{+}1,s)$-form is analogous) □

Lemma 11. Let J be a complex transverse structure on $(M, \mathcal{D})$. Then $\bar{\partial}\bar{\partial} = 0$ as a morphism $\mathcal{A}_B^{r,s} \to \mathcal{A}_B^{r,s+2}$.

Proof. First note that $d \circ d = 0$. This follows directly from our definition of d which mimics the invariant formula for the usual exterior derivative. From the definition of d and the fact that $[X, Y]^{0,1} = 0$ for all $X, Y \in \Gamma(TM^{1,0}, U)$, it is clear that $d(\mathcal{A}_B^{r,s}(U)) \subset \mathcal{A}_B^{r+1,s}(U) \oplus \mathcal{A}_B^{r,s+1}(U)$ for each open U. Then, $d = \bar{\partial} + \partial$ and since $d \circ d = 0$, we get $\bar{\partial}\bar{\partial} = \partial\partial = \bar{\partial}\partial + \partial\bar{\partial} = 0$ by comparing degrees. □

Definition 10. Given a manifold M and an almost complex transverse structure $(J, \mathcal{D})$ on M, we let the sheaf of *basic holomorphic forms* be $\Omega_B^r(U) := \mathcal{A}_B^{r,0}(U) \cap \ker(\bar{\partial})(U)$.

Definition 11. The (r,s)-th basic Dolbeault cohomology group of M with respect to an involutive distribution $\mathcal{D}$ and a complex transverse structure J is the s-th cohomology group of the complex $(\mathcal{A}_B^{r,\bullet}, \bar{\partial}^\bullet)$, denoted $H_B^{r,s}(M)$. (The complex transverse structure will always be understood from the context).

Lemma 12. Let (M, J) be an almost complex manifold, then J descends to an endomorphism of $TM/\mathcal{D}_J^\infty$ which is a transverse complex structure for $(M, \mathcal{D}_J^\infty)$ (c.f. Definition 5).

Proof. Since $\mathcal{D}_J^\infty = (TM^{1,0})_{\mathbb{R}}^\infty$, Lemma 6 implies that i) $J\mathcal{D}_J^\infty \subset \mathcal{D}_J^\infty$, ii) $\operatorname{Im} N_J \subset \mathcal{D}_J^\infty$ and iii) $(\mathcal{L}_X J)(Y) \in \mathcal{D}_J^\infty$ for all $X \in \mathcal{D}_J^\infty$, $Y \in \Gamma(TM)$. Then i) implies that J descends to the quotient $TM/\mathcal{D}_J^\infty$, and ii-iii) imply that J is a transverse complex structure for $\mathcal{D}_J^\infty$. □

Definition 12. Let (M, J) be an almost complex manifold. Since J is a complex transverse structure for $\mathcal{D}_J^\infty$, we have induced basic Dolbeault cohomology groups $H_B^{r,s}(M)$. These are the *basic Dolbeault Cohomology groups of* (M, J).

Remark 8. In the setting of the previous definition, it is not hard to see that $H_B^{r,s}(M)$ is canonically isomorphic to the "transverse Dolbeault Cohomology" of CGG [4]. This is what we meant by CGG's theory being a special case of E.K. Alaoui's theory, once extended to non-regular distributions.

4.2 Harmonic theory for basic (r, s)-forms

Just like in complex geometry, transverse almost complex structures are best behaved under compatibility conditions with a Riemannian metric. We record in this section and the next some of the work of E.K. Alaoui [13] on such structures for later use.

Recall that a Riemannian foliation on a smooth manifold M is the data of a pair $(\mathcal{F}, g_M)$ where $\mathcal{F}$ is an involutive *regular* distribution[7] together with a metric g_M on M which is bundle-like. This means that the metric g_Q on $Q = TM/\mathcal{F}$ induced by g_M is holonomy invariant :

$$\mathcal{L}_Y g_Q(X_1, X_2) = Y(g_Q(X_1, X_2)) - g_Q([Y, X_1], X_2) - g_Q(X_1, [Y, X_2]) = 0$$

where $Y \in \Gamma(\mathcal{F})$ and $X_1, X_2 \in \Gamma(Q)$. As a convention, we let $p := \operatorname{rank}_{\mathbb{R}} \mathcal{F}$ and $q := \dim M - p$. The foliation $\mathcal{F}$ is *transversely orientable* if there exists a nowhere vanishing smooth global section of $\wedge^q Q^*$. If M is oriented and $\mathcal{F}$ is transversely oriented, we define the characteristic form $\chi_{\mathcal{F}}$ to be the global form locally given by $e_1 \wedge \cdots \wedge e_p$ where $e_1, ..., e_p, e_{p+1}, ..., e_{p+q}$ is an oriented orthonormal co-frame adapted to $\mathcal{F}$. The harmonic theory of basic forms is best behaved when the leaves of the foliation are minimal in a sense which we define now. Let ∇^M be the Levi-Civita connection of (M, g_M) and $E_1, \ldots, E_p, E_{p+1} \ldots, E_{p+q}$ a local orthonormal frame adapted to $\mathcal{F}$. We let the *mean curvature vector field* be locally given by

$$H := \sum_{i=p+1}^{q} (\nabla^M_{E_i} E_i)^{\perp g_M}$$

and the *mean curvature of* $\mathcal{F}$ is defined as the 1-form $\kappa(X) := H^\flat(X) = g_Q(H, X)$. A foliation is *minimal* if the mean curvature vanishes. This condition on the leaves will become important to us because of the following.

Proposition 4 (see [28], 4.26)**.** Let $(M, \mathcal{F}, g)$ be a Riemannian foliation as above. The p+1-form $\varphi_0 := \kappa \wedge \chi_{\mathcal{F}} + d\chi_{\mathcal{F}}$ is such that $\iota_{X_1}\iota_{X_2} \ldots \iota_{X_p}\varphi_0 = 0$ for all $X_i \in \Gamma(\mathcal{F})$.

Let M be oriented and closed[8], and $\mathcal{F}$ be transversely oriented. Over $\mathcal{A}_B^\bullet$, we have a Hodge star $\bar{*} : \mathcal{A}_B^r \to \mathcal{A}_B^{q-r}$ defined by the relation:

$$g_Q(\alpha, \beta) \operatorname{vol}_M = \alpha \wedge \bar{*}\beta \wedge \chi_{\mathcal{F}} \tag{9}$$

[7] We use the letter $\mathcal{F}$ to follow the standard notation of the theory of Riemannian foliations (see e.g. [28],[13]).

[8] This means compact and boundaryless.

We define on $\mathcal{A}_B^\bullet$ the following inner product:

$$\langle \alpha, \beta \rangle_B := \int_M \alpha \wedge \bar{*}\beta \wedge \chi_{\mathcal{F}}$$

for $\beta, \alpha \in \mathcal{A}_B^r$. It is not hard to check that $\bar{*}\beta = (-1)^{p(q-r)} * (\beta \wedge \chi_{\mathcal{F}})$, where $*$ is the usual Hodge-star operator. The restriction d_B of d to basic forms then admits a formal adjoint:

Theorem 5 (see [28], theorem 7.10)**.** The formal adjoint of d_B in $\mathcal{A}_B^\bullet$ with respect to $\langle \_, \_ \rangle_B$ is given by

$$\delta_B \beta := (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa\wedge)\bar{*}\beta \tag{10}$$

where $\kappa\wedge$ is to be understood as the operator $\omega \mapsto \kappa \wedge \omega$.

Then the basic Laplacian $\Delta_B : \mathcal{A}_B^\bullet \to \mathcal{A}_B^\bullet$ is defined by $\Delta_B := \delta_B d_B + d_B \delta_B$. Letting $\mathcal{H}_B^\bullet := \ker \Delta_B$, the following Hodge decomposition holds:

Theorem 6 (see [28], theorem 7.22)**.** Let $(\mathcal{F}, g_M)$ be a transversely oriented Riemannian foliation on a closed oriented manifold M. Assume that $\kappa \in \Gamma(\mathcal{A}_B^1)$. Then there is a decomposition into mutually orthogonal subspaces

$$\mathcal{A}_B^r \cong \operatorname{im} d_B \oplus \mathcal{H}_B^r \oplus \operatorname{im} \delta_B$$

with finite-dimensional $\mathcal{H}_B^r$.

A *Hermitian foliation* is a triple $(\mathcal{F}, g_M, J)$ such that $(\mathcal{F}, g_M)$ is a Riemannian foliation, J a transverse complex structure on $\mathcal{F}$ and g_Q is compatible with J, meaning $g_Q(J\_, J\_) = g_Q(\_, \_)$. Fix a Hermitian foliation $\mathcal{F}$ on M. We again let $p = \operatorname{rank} \mathcal{F}$ and $q := \dim M - p$. Notice that since Q is naturally isomorphic to the normal bundle $\nu\mathcal{F}$ of $\mathcal{F}$, basic forms can be understood as holonomy-invariant sections of $\bigwedge^\bullet(\nu\mathcal{F})^*$.

Now let M be oriented. Since $(M, \mathcal{F}, g_M, J)$ is Hermitian, $\nu\mathcal{F}$ is orientable. A choice of orientation on $\nu\mathcal{F}$ then determines a characteristic form $\chi_{\mathcal{F}}$ on $\mathcal{F}$. We let $(\_, \_)$ denote the Hermitian product induced by g_Q on $Q_{\mathbb{C}} \cong (\nu\mathcal{F})_{\mathbb{C}}$. Since $\nu\mathcal{F}$ admits an endomorphism J squaring to $-\mathrm{Id}$, $\dim \nu\mathcal{F}$ is even, and we let $2t = \dim \nu\mathcal{F}$. Letting $*$ be the usual Hodge star operator with respect to g_M, we may define a $\mathbb{C}$-linear operator $\bar{*} : \mathcal{A}_B^{r,s} \to \mathcal{A}_B^{t-r,t-s}$ by the relation

$$\alpha \wedge \bar{*}\bar{\beta} = (\alpha, \beta) * \chi_{\mathcal{F}} \quad \text{where } \alpha, \beta \in \Omega_B^{r,s}. \tag{11}$$

It is not hard to see that $\bar{*}$ is just the $\mathbb{C}$-linear extension of the real star $\bar{*}$ defined above. It is clear that $\bar{*}\bar{*}\beta = (-1)^{r+s}\beta$ for any $\beta \in \Omega_B^{r,s}$. We let $d_B : \Omega_B^\bullet \to \Omega_B^{\bullet+1}$ be the restriction of the usual exterior derivative to basic forms. Analogously, we let ∂_B (resp. $\bar{\partial}_B$) be the restriction of $\partial : \Omega_Q^{r,s} \to \Omega_Q^{r+1,s}$ (resp. $\bar{\partial}$) to the space of basic (r,s)-forms. Define a Hermitian inner product on $\Omega_B^\bullet$ by

$$(\alpha, \beta)_B := \int_M (\alpha, \beta) \operatorname{Vol}_M, \quad \text{where } \alpha, \beta \in \Omega_B^p(M). \tag{12}$$

Note that equation (11) implies $(\alpha, \beta)_B = \int_M \alpha \wedge \bar{*}\bar{\beta} \wedge \chi_{\mathcal{F}}$.

Proposition 5 (compare with E.K. Alaoui [13], p.93)**.** Let M be a closed smooth manifold and $(M, \mathcal{F}, g_M, J)$ be a Hermitian foliation with minimal leaves[9] of codimension q and dimension p. The operator

$$\bar{\partial}_B^* := -\bar{*}\partial_B\bar{*} : \mathcal{A}_B^{r,s} \to \mathcal{A}_B^{r,s-1} \quad (\text{resp. } \partial_B^* := -\bar{*}\bar{\partial}_B\bar{*})$$

is the formal adjoint of $\bar{\partial}_B$ (resp. ∂_B) under $(\_,\_)_B$.

Proof. [10] Let $\alpha \in \mathcal{A}_B^{r,s-1}$ and $\beta \in \mathcal{A}_B^{r,s}$. Then

$$(\bar{\partial}_B\alpha, \beta)_B = \int_M \bar{\partial}_B\alpha \wedge \bar{*}\bar{\beta} \wedge \chi_{\mathcal{F}}. \tag{13}$$

Note that $\partial_B(\alpha \wedge \bar{*}\bar{\beta}) = 0$ since $\alpha \wedge \bar{*}\bar{\beta}$ is a $(t, t-1)$-form[11] So $d_B(\alpha \wedge \bar{*}\bar{\beta}) = \bar{\partial}_B(\alpha \wedge \bar{*}\bar{\beta})$. Then

$$d(\alpha \wedge \bar{*}\bar{\beta} \wedge \chi_{\mathcal{F}}) = \bar{\partial}_B(\alpha \wedge \bar{*}\bar{\beta}) \wedge \chi_{\mathcal{F}} + (-1)^{q-1}\alpha \wedge \bar{*}\bar{\beta} \wedge d\chi_{\mathcal{F}}.$$

It follows from Proposition 4 that $\alpha \wedge \bar{*}\bar{\beta} \wedge d\chi_{\mathcal{F}} = 0$. Indeed, it is a top-form, so it does not vanish if and only if it does not vanish on a co-frame $e^1, .., e^p, e^{p+1}, .., e^{p+q}$ adapted to $\mathcal{F}$. But since $\iota_X\beta = \iota_X\alpha = 0$ for all $X \in \Gamma\mathcal{F}$ and $d\chi_{\mathcal{F}} = \varphi_0$ vanishes on all p-tuples of sections of $\mathcal{F}$, $\alpha \wedge \bar{*}\bar{\beta} \wedge d\chi$ vanishes on any such co-frame. The Leibniz rule implies

$$d(\alpha \wedge \bar{*}\bar{\beta} \wedge \chi_{\mathcal{F}}) = \bar{\partial}_B\alpha \wedge \bar{*}\bar{\beta} \wedge \chi_{\mathcal{F}} + (-1)^{r+s-1}\alpha \wedge \bar{\partial}_B\bar{*}\bar{\beta} \wedge \chi_{\mathcal{F}} \tag{14}$$

Substituting (14) into (13) and applying Stoke's theorem, we get that

$$(\bar{\partial}_B\alpha, \beta)_B = \int_M (-1)^{r+s}\alpha \wedge \bar{\partial}_B\bar{*}\bar{\beta} \wedge \chi_{\mathcal{F}}. \tag{15}$$

Finally, substituting $(-1)^{r+s+1}\bar{*}(\bar{*}\bar{\partial}_B\bar{*})\bar{\beta} = \bar{\partial}_B\bar{*}\bar{\beta}$ in (15) gives

$$(\bar{\partial}_B\alpha, \beta)_B = (\alpha, -\bar{*}\partial_B\bar{*}\beta)_B = (\alpha, \bar{\partial}_B^*\beta)_B. \qquad \square$$

We let the $\bar{\partial}_B$-Laplacian be defined as $\Delta_{\bar{\partial}_B} := \bar{\partial}_B\bar{\partial}_B^* + \bar{\partial}_B^*\bar{\partial}_B$, and the corresponding space of harmonic forms be $\mathcal{H}_B^{r,s} := \ker \Delta_{\bar{\partial}_B} \cap \mathcal{A}_B^{r,s}$. Then we have the following theorem:

Theorem 7 (E.K. Alaoui [13], theorem 3.3.3.)**.** Let M be a closed manifold and $(\mathcal{F}, g_M, J)$ be a Hermitian foliation on M with minimal leaves[12]. Fix a transverse orientation of $\mathcal{F}$. Then $\mathcal{H}_B^{r,s}$ is finite dimensional and $\mathcal{A}_B^{r,s} = \operatorname{Im}\bar{\partial}_B \oplus \mathcal{H}_B^{r,s} \oplus \operatorname{Im}\bar{\partial}_B^*$.

A little linear algebra (analogous to [11], Corollary 3.2.9.) then shows that the map sending a harmonic form ω to its cohomology class $[\omega]$ induces an isomorphism $\mathcal{H}_B^{r,s} \xrightarrow{\sim} H_B^{r,s}(M)$.

[9] Note that E.K. Alaoui asks that the leaves of $\mathcal{F}$ be "homologically orientable" instead of minimal. When M is compact and connected, these two conditions are equivalent by [28], Theorem 7.56 and [13], Theorem 3.2.8.

[10] We give a proof because our signs differ from the original paper [13], p.93 which had omited the proof. Correctness of the signs is not crucials to the theory.

[11] Recall that $2t = q = \operatorname{codim}\mathcal{F}$.

[12] Note that E.K. Alaoui asks that $\mathcal{F}$ be "homologically orientable" instead of minimal. When M is compact and connected, these two conditions are equivalent by [28], thm 7.56 and [13], thm 3.2.8.

4.3 Hodge Theory on basic $(r, s)-$forms

We recall a version of the Hodge decomposition theorem for foliations with minimal leaves. This was worked out first by E.K. Alaoui [13] (for a recent perspective, see [12]). Given a smooth manifold M and a Hermitian foliation $(\mathcal{F}, g_M, J)$, we say that the foliation $\mathcal{F}$ is *Kähler* if the form $\omega := g_Q(J\_, \_)$ is closed (i.e. $d\omega = 0$). In this case, one can work out Kähler identities similar to those in the complex geometry case (see [13], p.96) and prove:

Proposition 6 (see [13], prop.3.4.5.)**.** Let M be a closed manifold and $(\mathcal{F}, g_M, J)$ be a Kähler foliation on M with minimal leaves. After fixing a transverse orientation of $\mathcal{F}$. Then we have $\Delta_B = 2\Delta_{\bar{\partial}_B}$.

From which the following theorem follows:

Theorem 8 (see [13], prop.3.4.6.)**.** Let M be a closed manifold and let $(\mathcal{F}, g_M, J)$ be a Kähler foliation on M with minimal leaves. Fix a transverse orientation of $\mathcal{F}$. Given a Δ_B-harmonic basic r-form α, the projections $\Pi^{r,s}\alpha \in \mathcal{A}_B^{r,s}$ are harmonic as well. Then the theorems 7 and 6 imply the decomposition

$$H^k(M; \mathcal{F}) = \bigoplus_{r+s=k} H^{r,s}(M, \mathcal{F}).$$

4.4 Transversaly Kähler manifolds

In an ideal world, any compact almost Kähler manifold (M, J, g) would be such that $(\mathcal{D}_J^\infty, J, g)$ is a Kähler foliation and we could apply the Hodge theoretical tools of subsections 4.2 and 4.3 on it, but it is unfortunately not the case. In fact, we have the following characterization of those almost Kähler manifolds such that $\mathcal{D}_J^\infty$ is a Kähler foliation:

Proposition 7. Let (M, g, J) be an almost Kähler manifold of constant type with $\omega := g(J\_, \_)$. The following statements are equivalent:

1. $(\mathcal{D}_J^\infty, J, g)$ is a Kähler foliation.
2. $\mathcal{L}_X g(Y, Z) = 0$ for all $X \in \mathcal{D}_J^\infty$, $Y, Z \in \nu\mathcal{D}_J^\infty$.
3. $\mathcal{L}_X \omega(Y, Z) = 0$ for all $X \in \mathcal{D}_J^\infty$, $Y, Z \in \nu\mathcal{D}_J^\infty$.
4. $\nu\mathcal{D}_J^\infty$ is involutive.

Proof. (1) $\Leftrightarrow$ (2). Immediate from the definition of Kähler foliation.
(2) $\Leftrightarrow$ (3). Let $X \in \mathcal{D}_J^\infty$, $Y, Z \in \nu\mathcal{D}_J^\infty$ and compute:

$$\begin{aligned}
\mathcal{L}_X \omega(Y, Z) &= X(\omega(Y, Z)) - \omega([X, Y], Z) - \omega(Y, [X, Z]) \\
&= X(g(JY, Z)) - g(J[X, Y], Z) - g(JY, [X, Z]) \\
&\qquad\qquad + \big(g([X, JY], Z) - g([X, JY], Z)\big) \\
&= \mathcal{L}_X g(JY, Z) + g([X, JY] - J[X, Y], Z) \\
&= \mathcal{L}_X g(JY, Z) + g((\mathcal{L}_X J)Y, Z) \qquad (\dagger)
\end{aligned}$$

Then, by definition of $\mathcal{D}_J^\infty$, one has $(\mathcal{L}_X J)Y \in \mathcal{D}_J^\infty$ for all $X \in \mathcal{D}_J^\infty$, so that $g((\mathcal{L}_X J)Y, Z) = 0$, since $Z \in \nu\mathcal{D}_J^\infty$. Note also that $J\,\nu\mathcal{D}_J^\infty \subset \nu\mathcal{D}_J^\infty$, since

$$g(X, JY) = g(JX, -Y) = 0 \quad \text{for all } X \in \mathcal{D}_J^\infty \text{ and } Y \in \nu\mathcal{D}_J^\infty.$$

The equality (†) then gives $\mathcal{L}_X\omega(Y,Z)=0$ for all $Y,Z\in\nu\mathcal{D}_J^\infty$ if and only if $\mathcal{L}_X g(Y,Z)=0$ for all $Y,Z\in\nu\mathcal{D}_J^\infty$, yielding the conclusion.
(3) $\Rightarrow$ (4). Since $d\omega=0$, Cartan's formula implies

$$\begin{aligned}0=\mathcal{L}_X\omega(Y,Z)&=d(\iota_X\omega)(Y,Z)\\&=Y\omega(X,Z)-Z\omega(X,Y)-\omega(X,[Y,Z])\\&=-\omega(X,[Y,Z])\\&=-g(JX,[Y,Z])\end{aligned}$$

Since this holds for all $X\in\mathcal{D}_J^\infty$, we get $[Y,Z]\in\nu\mathcal{D}_J^\infty$, i.e. $\nu\mathcal{D}_J^\infty$ is involutive.
(4) $\Rightarrow$ (3). Let X,Y,Z be as above. Involutivity of $\nu\mathcal{D}_J^\infty$ implies that $Y\omega(X,Z)=Z\omega(X,Y)=\omega([Y,Z],X)=0$, and thus

$$\begin{aligned}0=d\omega(X,Y,Z)&=X\omega(Y,Z)-\omega([X,Y],Z)+\omega(Y,[X,Z])\\&=\mathcal{L}_X\omega(Y,Z)\end{aligned}$$

□

Definition 13. We say that a closed almost Kähler manifold (M,J,g) of constant type is *transversely Kähler* if $\nu\mathcal{D}_J^\infty$ is involutive. A *minimal* transversely Kähler manifold is a transversely Kähler manifold such that $\mathcal{D}_J^\infty$ is a minimal foliation.

Example 3. Any closed Kähler manifold is a minimal transversely Kähler manifold, since in this case, $\nu\mathcal{D}_J^\infty=TM$.

Example 4 (Kodaira-Thurston Torus)**.** We denote by H, the 3-dimensional Heisenberg Lie group, defined as the group of matrices of the form

$$H=\left\{\begin{pmatrix}1&b&c\\0&1&a\\0&0&1\end{pmatrix}\;\middle|\;a,b,c\in\mathbb{R}\right\}\tag{†}$$

we define the Kodaira-Thurston as $\mathrm{KT}:=H\times\mathbb{R}/H_\mathbb{Z}\times\mathbb{Z}$. The space $H\times\mathbb{R}$ comes with a group structure given by

$$\left(\begin{array}{c|c}A&0\\\hline 0&d\end{array}\right)\cdot\left(\begin{array}{c|c}A'&0\\\hline 0&d'\end{array}\right)=\left(\begin{array}{c|c}AA'&0\\\hline 0&d+d'\end{array}\right).\tag{††}$$

The space $H_\mathbb{Z}\times\mathbb{Z}\subset H\times\mathbb{R}$ is the subgroup of $H\times\mathbb{R}$ for which $a,b,c,d\in\mathbb{Z}$. $H\times\mathbb{R}$ is endowed with the Euclidean topology, so $H_\mathbb{Z}\times\mathbb{Z}$ is a discrete subgroup of $H\times\mathbb{R}$. The Lie algebra of $H\times\mathbb{R}$ has a basis of four elements $X_1,X_2,X_3,X_4\in\mathrm{Lie}(H\times\mathbb{R})$ subject to the relations

$$[X_i,X_j]=\begin{cases}X_3&\text{if } i=1 \text{ and } j=2\\0&\text{otherwise}\end{cases}$$

Since $H_\mathbb{Z}\times\mathbb{Z}$ is discrete, these vector fields descend to a global frame on KT. We define a left-invariant almost complex structure on KT by letting

$$JX_1:=-X_4\quad\text{and}\quad JX_3:=-X_2.$$

A straightforward verification shows that Im $N_J=\langle X_3,X_2\rangle$ and $\mathcal{D}_J^\infty=\mathrm{Im}N_J$, so that (KT,J) has constant type 1. We let $x_1,\dots,x_4\in\mathrm{Lie}(H\times\mathbb{R})^*$

be the left-invariant dual basis of $X_1, ..., X_4$. The x_i's again descend to a global coframe on KT . It is easy to check that $dx_1 = dx_2 = dx_4 = 0$ and $dx_3 = x_2 \wedge x_1$. We let $\omega := x_1 \wedge x_4 + x_2 \wedge x_3$. A straight forward computation shows $d\omega = 0$ and $\omega(J\_, J\_) = \omega(\_, \_)$ so (KT, J, ω) is an almost Kähler manifold. The induced compatible metric g is given by $g = \sum_{i=1}^{4} x_i \otimes x_i$ and thus the orthogonal complement $\nu\mathcal{D}_J^\infty$ of $\mathcal{D}_J^\infty$ is given by $\langle X_1, X_4\rangle$. Since $[X_1, X_4] = 0$, $\nu\mathcal{D}_J^\infty$ is involutive and thus (KT, g, J) is transversely Kähler. Recall that the mean curvature of a leaf of $\mathcal{D}_J^\infty$ is given by

$$H = \left(\nabla_{X_2} X_2 + \nabla_{X_3} X_3\right)^\perp,$$

where $(\cdot)^\perp$ denotes the projection onto the normal bundle $\nu\mathcal{D}_J^\infty$. For a left-invariant orthonormal frame, the Koszul formula reads

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle.$$

Since the only nonzero bracket is $[X_1, X_2] = X_3$, we compute:

$$2\langle \nabla_{X_2} X_2, Z\rangle = -\langle [X_2, Z], X_2\rangle + \langle [Z, X_2], X_2\rangle = 0,$$

for all Z, hence $\nabla_{X_2} X_2 = 0$. We compute similarly that $\nabla_{X_3} X_3 = 0$ and thus the mean curvature vanishes, implying that KT is a minimal transversely Kähler manifold.

We now study the leaf space of the foliation associated with $\mathcal{D}_J^\infty$. Let e be the identity matrix and note that the Lie algebra $\mathrm{Lie}(H \times \mathbb{R}) \cong T_e(H \times \mathbb{R})$ is explicitly given as the subspace of $M(4, \mathbb{R})$ spanned by

$$X_1 = \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \text{ and } X_4 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}.$$

Then $\mathcal{D}_J^\infty$ is the subspace of $T(H \times \mathbb{R})$ given by left translations of the subspace

$$(\mathcal{D}_J^\infty)_e = \left\langle \begin{pmatrix} 0&0&b&0\\0&0&c&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \ :\ b, c \in \mathbb{R} \right\rangle \subset T_e(KT).$$

Given $A \in H \times \mathbb{R}$, we let $[A] \in KT$ be the projection onto KT. It is straightforward to verify that the leaf of $\mathcal{D}_J^\infty$ through $[A]$ is given by $L_{[A]} = \{[A + V] \ :\ V \in (\mathcal{D}_J^\infty)_e\} \subset KT$. We claim that the leaf space is a 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. To see this, it suffices to show that the map

$$\begin{aligned} \pi : &\mathrm{KT} \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 \\ &\left[\begin{pmatrix} 1&a&c&0\\0&1&b&0\\0&0&1&0\\0&0&0&d \end{pmatrix}\right] \mapsto [(a, d)] \end{aligned} \tag{16}$$

is a well defined submmersion with fibers equal to the leaves of $\mathcal{D}_J^\infty$. The smoothness and submersion property of π follow at once from the fact that in local charts, π is simply the corresponding projection $H \times \mathbb{R} \to \mathbb{R}^2$ restricted to some open set of $H \times \mathbb{R}$. Surjectivity of π is clear. The fact that the fibers of π are exactly the leaves of $\mathcal{D}_J^\infty$ follows from observing that over $KT/\mathcal{D}_J^\infty$, one has

$$\begin{pmatrix} 1&a&c&0\\0&1&b&0\\0&0&1&0\\0&0&0&d \end{pmatrix} \sim \begin{pmatrix} 1&a'&c'&0\\0&1&b'&0\\0&0&1&0\\0&0&0&d' \end{pmatrix} \quad \text{if and only if } a - a' \in \mathbb{Z} \text{ and } d - d' \in \mathbb{Z}.$$

Note that since KT is compact, the fibres of π are compact and thus the leaves of $\mathcal{D}_J^\infty$ are compact.

Example 5. Any almost Kähler manifold of constant type 0 is transversely Kähler (because $\nu\mathcal{D}_J^\infty = 0$). By [19], Corollary 4.3., this includes nearly Kähler manifolds of dimension 6 that are not Kähler. However their "transverse complex geometry" is trivial since they have no local $J-$holomorphic functions and their higher Basic Dolbeault cohomology groups vanish.

4.5 Basic Dolbeault Cohomology and Cirici-Wilson's Dolbeault cohomology

In 2018, Cirici-Wilson introduced a Dolbeault cohomology $H^{r,s}_{\mathrm{Dol}}(M)$ on an almost complex manifold (M, J) which does not require restricting to basic forms. Cahen-Gutt-Gutt observed in [4] Proposition 3.8. that $H^{r,0}_B(M) \cong H^{r,0}_{\mathrm{Dol}}(M)$ for all $r \in \mathbb{N}$. We show that on a minimal transversely Kähler manifold M, there is an *injective* morphism

$$H^{r,s}_B(M) \hookrightarrow H^{r,s}_{\mathrm{Dol}}(M)$$

for all $r, s \in \mathbb{N}$. There might be an injective morphism for more general almost complex structures. This morphism at least covers all explicit examples worked out in Cirici-Wilson's (and CGGs) paper.

Begin by recalling Cirici-Wilson's Dolbeault cohomology. Let (M^{2n}, J) be an almost complex manifold. The complexified de Rham algebra splits

$$\mathcal{A}^k(M) = \bigoplus_{r+s=k} \mathcal{A}^{r,s}(M),$$

and the exterior differential decomposes

$$d\omega = \overline{\mu}\omega + \overline{\partial}\omega + \partial\omega + \mu\omega \in \mathcal{A}^{r-1,s+2} \oplus \mathcal{A}^{r,s+1} \oplus \mathcal{A}^{r+1,s} \oplus \mathcal{A}^{r+2,s-1}$$

From $d^2 = 0$ and bidegree reasons, one obtains the relations

$$\overline{\mu}^2 = 0, \qquad \overline{\partial}\,\overline{\mu} + \overline{\mu}\,\overline{\partial} = 0, \quad \overline{\mu}\,\partial + \partial\,\overline{\mu} + \overline{\partial}^2 = 0,$$

and further relations involving ∂, μ (not needed below).

Definition 14 (Cirici–Wilson Dolbeault cohomology)**.** First take the $\overline{\mu}$–cohomology

$$H^{r,s}_{\overline{\mu}}(M) \;:=\; \frac{\ker\{\overline{\mu} : \mathcal{A}^{r,s} \to \mathcal{A}^{r-1,s+2}\}}{\operatorname{im}\{\overline{\mu} : \mathcal{A}^{r+1,s-2} \to \mathcal{A}^{r,s}\}}.$$

Because $\overline{\partial}\overline{\mu} + \overline{\mu}\overline{\partial} = 0$, the operator $\overline{\partial}$ descends to $H^{r,s}_{\overline{\mu}}(M)$, and by $\overline{\mu}\partial + \partial\overline{\mu} + \overline{\partial}^2 = 0$ the induced $\overline{\partial}$ squares to 0 there. Define

$$H^{r,s}_{\mathrm{Dol}}(M) \;:=\; H^q\big(H^{r,*}_{\overline{\mu}}(M), \overline{\partial}\big).$$

When J is integrable ($\overline{\mu} \equiv 0$), this recovers the classical Dolbeault cohomology.

There is a harmonic identity that will be important to us. For $\delta \in \{\bar{\mu}, \bar{\partial}, \partial, \mu\}$, write $\delta^* = -* \,\overline{\delta}\, *$ and $\Delta_\delta = \delta\delta^* + \delta^*\delta$. Let $H^{r,s}_\delta := \ker(\Delta_\delta) \cap \mathcal{A}^{r,s}$.

Proposition 8 (Cirici-Wilson [5], thm 4.18.)**.** There is a natural injection

$$H^{r,s}_{\overline{\partial}} \cap H^{r,s}_{\overline{\mu}} \;\hookrightarrow\; H^{r,s}_{\mathrm{Dol}}(M).$$

The idea now is to show that harmonic basic forms over minimal transversely Kähler manifolds belong to $H^{r,s}_{\bar\partial} \cap H^{r,s}_{\bar\mu}$.

Lemma 13. Let (M, J, g) be an almost Hermitian manifold. Every basic (r, s)-form is also $\bar\mu$-harmonic, i.e. $\mathcal{A}^{r,s}_B(M) \subset \mathcal{H}^{r,s}_{\bar\mu}(M)$

Proof. Let $\omega \in \mathcal{A}^{r,s}_B(M)$. Begin by observing that $\bar\mu\omega = 0$. Let $X_1, ..., X_{r+s}$ be vector fields, then $\bar\mu\omega(X_1, ..., X_{r+s+1})$ is given by

$$
\begin{aligned}
&d\omega(X_1^{1,0}, .., X_{r-1}^{1,0}, X_r^{0,1}, .., X_{r+s+1}^{0,1}) \\
&= \sum_{r\le i\le j\le r+s+1} (-1)^{i+j}\omega([X_i^{0,1}, X_j^{0,1}]^{0,1}, X_1^{1,0}...X_{r-1}^{1,0}, X_r^{0,1}...\widehat{X_{r+i}^{0,1}}...\widehat{X_{r+j}^{0,1}}...X_{r+s+1}^{0,1}),
\end{aligned}
$$

where we used the fact that ω is a (r, s)-form to simplify equation (8). For later use, it is worth noting that this equation implies that $\bar\mu$ is C^∞-linear. Let $X_i^{0,1} = Y_i + iJY_i$ where Y_i is real. We see from equation (2) that

$$[X_i^{1,0}, X_j^{1,0}]^{0,1} = \overline{N_J(X_i^{0,1}, X_j^{0,1})} = N_J(Y_i, Y_j) - iJN_J(Y_i, Y_j).$$

This belongs to $\mathcal{D}^\infty_J \otimes \mathbb{C}$. Since ω is basic, we have $\iota_{[X_i^{1,0}, X_j^{1,0}]^{0,1}}\omega = 0$, and thus $\bar\mu\omega = 0$.

We now show that $\bar\mu^*\omega = 0$. Since μ is $C^\infty-$linear, $\bar\mu^*$ and μ are fiberwise adjoints, i.e., $\langle\bar\mu^*\omega_x, \alpha_x\rangle_x = \langle\omega_x, \bar\mu\alpha_x\rangle_x$ for each $x \in M$ (and not just formal L^2-adjoints, see [5], Lemma 4.2. and Lemma 4.3. for a proof.) Let $E_1, ..., E_p, E_{p+1}, ..., E_{p+q}$ be an orthonormal basis of TM_x such that $E_1, ..., E_p$ span $(\mathcal{D}^\infty_J)_x$ and $E_{p+1}, .., E_{p+q}$ span the orthogonal complement $\nu(\mathcal{D}^\infty_J)_x$. Let $e_1, ..., e_{p+q}$ be the orthonormal dual basis. Since ω is basic, ω_x is a linear combination of elements of the form $e_{i_1} \wedge \cdots \wedge e_{i_{r+s}}$ where $e_{i_j} \in \nu(\mathcal{D}^\infty_J)^*_x$. Now let $\alpha \in \mathcal{A}^{r+1,s-2}(M)$, we claim that either $\bar\mu\alpha = 0$ or it is a linear combination of elements of the form $e_{i_1} \wedge \cdots \wedge e_{i_{r+s}}$ where at least one of the $e_{i_j} \in (\mathcal{D}^\infty_J)^*_x$. Observe that since $\bar\mu$ is $C^\infty-$linear, it is enough to show the statement for $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_{r+s}}$. Write α as $\alpha = e_B \wedge (e_{j_1} \wedge \cdots \wedge e_{j_k})$ where $e_B \in \wedge^\bullet\nu(\mathcal{D}^\infty_J)_x$ and $e_{j_i} \in (\mathcal{D}^\infty_J)^*$. Now

$$\bar\mu(e_B \wedge (e_{j_1} \wedge \cdots \wedge e_{j_{r+s}}) = \Sigma_{i=1}^k(-1)^{j-1}e_B \wedge \bar\mu e_{j_i} \wedge e_{j_1} \wedge .. \wedge \widehat{e_{j_i}} \wedge .. \wedge e_{j_{r+s}},$$

from which we see that either $\bar\mu e_{j_i} = 0$ for each e_{j_i}, in which case $\bar\mu\alpha = 0$ or $\bar\mu e_{i_j} \neq 0$ for some j, in which case $\bar\mu\alpha$ is a linear combination of basis elements containing at least one $e_{j_i} \in (\mathcal{D}^\infty_J)^*_x$. We conclude that for all $\alpha \in \mathcal{A}^{r+1,s-2}(M)$,

$$\langle\bar\mu^*\omega, \alpha\rangle = \langle\omega, \bar\mu\alpha\rangle = 0,$$

so that $\bar\mu^*\omega = 0$ and thus $\Delta_{\bar\mu}\omega = 0$. □

Proposition 9. Let (M, J, g) be a minimal transversely Kähler manifold. There exists an injection $H^{r,s}_B(M) \hookrightarrow H^{r,s}_{\mathrm{Dol}}(M)$ which sends harmonic forms to harmonic forms.

Proof. By Tondeur [28] §7.28, the d-Laplacian Δ agrees with the basic Laplacian Δ_B (Δ_B was defined in §4.2) and by proposition 6, $\Delta_B = 2\Delta_{\bar\partial_B}$, so that $\Delta_{\bar\partial} = \Delta_{\bar\partial_B}$. Hence if $\omega \in \mathcal{A}^{r,s}_B(M)$ is $\Delta_{\bar\partial_B}$-harmonic, it is both $\bar\mu$ and $\bar\partial$ harmonic, and it follows that we have inclusions

$$H^{r,s}_B(M) = \mathcal{H}^{r,s}_{\bar\partial_B}(M) \hookrightarrow \mathcal{H}^{r,s}_{\bar\partial} \cap \mathcal{H}^{r,s}_{\bar\mu} \hookrightarrow H^{r,s}_{\mathrm{Dol}}(M)$$

□

Remark 9. Regarding question 4 in the introduction, it would be interesting to see if the basic cohomology over $\nu\mathcal{D}_J^\infty$ (which exists over transversely Kähler manifolds since $\nu\mathcal{D}_J^\infty$ is involutive) also accounts for some of the cohomology classes in $H^{r,s}_{\mathrm{Dol}}(M)$

4.6 Dolbeault Isomorphism

We show how the basic cohomology of $\mathcal{O}_M^J$ relates to the transverse Dolbeault cohomology of (M, J).

Proposition 10. Let (M, J) be of constant type. Then the sequence

$$0 \to \Omega_B^r \to \mathcal{A}_B^{r,0} \xrightarrow{\bar\partial} \mathcal{A}_B^{r,1} \xrightarrow{\bar\partial} \cdots$$

is exact. (Recall that Ω_B^r was defined in Definition 10).

Proof. Fix $p \in M$, since (M, J) is of constant type, say of type m, there is a set of functionally independent J-holomorphic functions on a neighbourhood U of p, $f^1, ..., f^m$ such that any other J-holomorphic function is functionally dependent on them. We let $\pi : U \to \mathbb{C}^m$ be $\pi = (f^1, ..., f^m)$. Shrink U if necessary so that the fibres of π are connected, and let $V := \pi(U) \subset \mathbb{C}^m$. On V, we consider the usual sheaves of $(r, s)-$forms, $\mathcal{A}^{r,s}_{\mathbb{C}^m}|_V$. Assume $r = 0$ for now. The usual $\bar\partial-$Poincaré lemma (see [11]) is the statement that the following sequence:

$$0 \to \mathcal{O}_{\mathbb{C}^m}|_V \hookrightarrow \mathcal{A}^{0,0}_{\mathbb{C}^m}|_V \xrightarrow{\bar\partial} \mathcal{A}^{0,1}_{\mathbb{C}^m}|_V \xrightarrow{\bar\partial} \cdots \tag{17}$$

is exact. Since π is flat (Theorem 4), π^* is an exact functor and thus $\pi^*\mathcal{A}^{0,\bullet}_{\mathbb{C}^m}|_V$ is also exact. Since $\pi^{-1}\mathcal{O}_{\mathbb{C}^m}|_V \cong \mathcal{O}_M^J|_U$ (again Theorem 4), we have

$$\pi^*(\mathcal{O}_{\mathbb{C}^m}|_V) := \pi^{-1}\mathcal{O}_{\mathbb{C}^m}|_V \otimes_{\pi^{-1}\mathcal{O}_{\mathbb{C}^m}} \mathcal{O}_M^J|_U \cong \pi^{-1}\mathcal{O}_{\mathbb{C}^m}|_V \cong \mathcal{O}_M^J|_U.$$

We then claim that $\pi^*\mathcal{A}^{0,s}_{\mathbb{C}^m}|_V = \mathcal{A}^{0,s}_B|_U$. Indeed, since the fibres of π are connected, it induces an isomorphism $\pi^* : \mathcal{A}^\bullet|_U \to \mathcal{A}^\bullet_B|_V$ (see [23] Proposition 3). The inverse of π^* which we denote by π_* is defined by the relation

$$(\pi_*\omega)_{\pi(p)}(d\pi_p\, X_1, ..., d\pi_p\, X_n) = \omega_p(X_1, \ldots, X_n) \tag{18}$$

for all $p \in U$ and $X_i \in T_pM$. Observe that it suffices to check that π^* and π_* sends $(0, s)$-forms to $(0, s)$-forms to conclude. By Theorem 4 we have the equality $(\mathcal{D}_J^\infty)_p = \ker d\pi_p$ and thus π induces an isomorphism $d\pi : Q|_U := TM/\mathcal{D}_J^\infty|_U \to T\mathbb{C}^m|_V$. The fact that π is $(J, i)-$holomorphic (again Theorem 4) implies that π^* sends (r, s) form to (r, s) form (this follows from $d\pi$ inducing isomorphisms $Q^{1,0} \xrightarrow{\sim} (T\mathbb{C}^m)^{1,0}$ and $Q^{0,1} \xrightarrow{\sim} (T\mathbb{C}^m)^{0,1}$). It follows immediately from (18) that π_* sends (r, s)-forms to (r, s)-forms as well.

Now applying the exact functor π^* to the sequence (17) yields an exact sequence:

$$0 \;\to\; \mathcal{O}_M^J|_U \;\to\; \mathcal{A}^{0,0}_B|_U \xrightarrow{\bar\partial} \mathcal{A}^{0,1}_B|_U \xrightarrow{\bar\partial} \cdots .$$

The general case, where $r \neq 0$, is handled similarly. We first apply the functor $\otimes_{\mathcal{O}_{\mathbb{C}^m}} \Omega^r_{\mathbb{C}^m}|_V$ to the sequence (17) and then apply π^* to conclude that

$$0 \;\to\; \Omega^r_B|_U \xrightarrow{i} \mathcal{A}^{r,0}_B|_U \xrightarrow{\bar\partial} \mathcal{A}^{r,1}_B|_U \xrightarrow{\bar\partial} \mathcal{A}^{r,2}_B|_U \xrightarrow{\bar\partial} \cdots$$

is a resolution of Ω^r_B. □

Remark 10 (Dolbeault Isomorphism for constant type ACS)**.** Let (M, J) be an almost complex manifold of constant type and suppose the sheaves $\mathcal{A}_B^{r,s}$ are acyclic. Then Proposition 10 gives a resolution of $\mathcal{O}_M^J$ and it follows that $H^s(M, \Omega_B^r) = H_B^{r,s}(M)$.

Unfortunately, $\mathcal{A}_B^{r,s}$ need not be acyclic in general, as the following example shows:

Example 6. Take any almost complex manifold (M, J) such that $\mathcal{D}_J^\infty = TM$ (e.g., any strictly nearly Kähler manifold of dimension 6, see [19], Corollary 4.3.) with some $H^i(M, \mathbb{C}) \neq 0$ (specifically, one could take $SU(2) \times SU(2)$ with the nearly Kähler structure of Example 5.12 in [5]). Then $\mathcal{A}_B^{0,0} = \underline{\mathbb{C}}$ which is not acyclic, since $H^{i \geq 0}(M, \mathbb{C}) \neq 0$.

It is still possible to obtain a geometrically meaningful version of Corollary 2 without any acyclicity assumption. To do so, we restrict our sheaves to a coarser topology of M.

Definition 15 (see also [18], §1.5.)**.** Let $(M, \mathcal{F})$ be a foliated manifold and $\pi : M \to M/\mathcal{F}$ the projection onto the leaf space ($M/\mathcal{F}$ is endowed with the quotient topology). The *saturated topology* of M is the topology given by the open sets of the form $\pi^{-1}(U)$ where U is open in $M/\mathcal{F}$. A *saturated open cover* is an open cover of M_{sat}.

Every sheaf we considered on M with the manifold topology restricts to a sheaf on M with the saturated topology. When we want to denote the use of the saturated topology explicitly, we write M_{sat}. For example $H^\bullet(M_{\mathrm{sat}}, \mathcal{G})$ is the sheaf cohomology of the sheaf $\mathcal{G}$ over M_{sat}. The proof of the following Poincaré lemma is an extension of the proof of Proposition 2.6 in [2] to the complex case.

Lemma 14 (Basic $\bar{\partial}-$lemma for the saturated topology)**.** Let $(M, \mathcal{F}, g, J)$ be a Hermitian foliation on a smooth manifold M with *compact* leaves. Then the sequence

$$\cdots \to \mathcal{A}_B^{r,s-1} \xrightarrow{\bar{\partial}} \mathcal{A}_B^{r,s} \xrightarrow{\bar{\partial}} \mathcal{A}_B^{r,s+1} \to \cdots$$

is exact as a sequence of sheaves over M_{sat}.

Proof. Let $x \in M$, we will show there exists a small enough neighbourhood of x in M_{sat} on which $\bar{\partial}\omega = 0$ implies $\omega = \bar{\partial}\eta$. Let L be the leaf containing x and U an open ϵ-tubular neighbourhood of L. By the general theory of Riemannian foliations (see Molino [18], Lemma 3.7), we may choose ϵ small enough so that U is a saturated open set. Since L is compact we may also choose ϵ small enough so that if $V = \{(x, v) \in \nu TL \ : \ |v|_g \leq \epsilon\}$ then $\exp : V \to U$ is a diffeomorphism ([15], Theorem 5.25). Let $B_\epsilon \subset \nu TL_x$ be the ball of radius ϵ centered at 0, then $D := \exp(B_\epsilon)$ is transversal to each leaf in U. Compactness of L implies that its holonomy group Γ is finite; thus, taking ϵ smaller if necessary, we can assume that Γ is a group of diffeomorphisms of D. We can also assume that D is contained in a flat chart. Then, by choosing a local frame of TM adapted to $T\mathcal{F}$ and TD, we easily find a projection[13] $p : TM|_D \to TD$ such that the sequence

$$0 \to T\mathcal{F}|_D \to TM|_D \xrightarrow{p} TD \to 0 \qquad (\dagger)$$

[13]For example, since D is in a flat chart, we may choose $\partial_1, ..., \partial_k$ spanning $T\mathcal{F}|_D$ and find $E_1, ..., E_s$ spanning TD. The union is a local frame of TM by transversality. Then choose π to be the bundle map $a^i\partial_i + b^jE_j \mapsto b^jE_j$.

is exact and $j : D \hookrightarrow M$ induces a section $dj : TD \to TM|_D$ of p. Let $\pi : TM|_D \to TM|_D/T\mathcal{F}|_D =: Q|_D$ be the projection, then we have an induced isomorphism $\pi \circ dj : TD \cong Q|_D$. We get an induced almost complex structure $\hat{J}$ on TD by pulling back $J : Q \to Q$ by $\pi \circ dj$. We claim that $\hat{J}$ is integrable. Let $X, Y \in \Gamma(TD)$, note that $N_{\hat{J}}(X,Y) = 0$ if and only if $\pi\, dj\, \big(N_{\hat{J}}(X,Y)\big) = 0$. Let $\tilde{X}$ and $\tilde{Y}$ be any extensions of X and Y to $TM|_U$. Then $dj_p X_p = \tilde{X}_{j(p)}$ (similarly for Y) and we see easily that for each $p \in D$, $\pi_p\, dj_p\, \big(N_{\hat{J}}(X,Y)_p\big) = N_J(\pi\tilde{X}, \pi\tilde{Y})_p = 0$ ($N_J = 0$ since J is a transverse complex structure[14].) Denote by $\Omega^\bullet(D)^\Gamma$ the space of smooth Γ-invariant forms on D. We claim that $j : D \to M$ induces an isomorphism $\mathcal{A}^{r,s}_B(U) \to \mathcal{A}^{r,s}(D) \cap \Omega^{r+s}(D)^\Gamma$. The fact that $j^* : \Omega^\bullet_B(U) \to \Omega^\bullet(D)^\Gamma$ is an isomorphism is best seen from Satake's theory of V-manifolds. Assuming V-manifold theory (see [24]), the argument is as follows: by Molino's structure theorem ([18], Theorem 3.7.) (and its proof) we have the following commutative diagram of V-manifolds inducing a commutative diagram at the level of differential forms:

$$\begin{tikzcd} U \arrow[d, two heads, "\pi_U"'] & D \arrow[l, hook', "j"'] \arrow[d, two heads, "\pi_D"] \\ U/\mathcal{F}|_U & D/\Gamma \arrow[l, hook', two heads, "{[j]}"] \end{tikzcd} \qquad \begin{tikzcd} \Omega^\bullet(U) \arrow[r, two heads, "j^*"] & \Omega^\bullet(D) \\ \Omega^\bullet(U/\mathcal{F}|_U) \arrow[u, hook, "\pi_U^*"'] \arrow[r, hook, two heads, "{[j]^*}"'] & \Omega^\bullet(D/\Gamma) \arrow[u, hook, "\pi_D^*"'] \end{tikzcd}$$

where $[j]$ is an isomorphism and π_U and π_D are submersions (of V-manifolds). Then $\pi_U^*(\Omega^\bullet(U/\mathcal{F}|_U)) = \Omega^\bullet_B(U)$, $\pi_D^*\Omega^\bullet(D/\Gamma) = \Omega^\bullet(D)^\Gamma$ and the fact that $[j]$ is an isomorphism implies that $j^* : \Omega^\bullet_B(U) \to \Omega^\bullet(D)^\Gamma$ is an isomorphism as well. Now complexify the sequence (†) and observe that for j^* to send (r,s) forms to (r,s) forms, it suffice to show that $\pi \circ dj : TD \to Q$ satisfies $\hat{J} \circ (\pi \circ dj) = (\pi \circ dj) \circ J$ which is the case by definition of $\hat{J}$. Let us denote by $\psi : \mathcal{A}^{r,s}(D) \cap \Omega^{r+s}(D)^\Gamma \to \mathcal{A}^{r,s}_B(U)$ the inverse of j^*.

Choose $\omega \in \mathcal{A}^{r,s}_B$ such that $\bar{\partial}\omega = 0$. Then $\bar{\partial} j^*\omega = 0$ and since D is a complex manifold, the usual $\bar{\partial}-$lemma implies that $j^*\omega = \bar{\partial}\sigma$ where $\sigma \in \mathcal{A}^{r,s-1}(U)$. Since $\mathcal{F}$ is Hermitian, Γ acts on D by holomorphic isometries (see [29]) and hence, $\sigma_\Gamma := \frac{1}{|\Gamma|}\sum_{\xi \in \Gamma} \xi^*\sigma$ is a (r,s)-form which is Γ-invariant. Then $\bar{\partial}\sigma_\Gamma = \frac{1}{|\Gamma|}\sum_{\xi\in\Gamma} \xi^* j^*\omega$. Since ω is basic, it is the pullback of a form on $U/\mathcal{F}|_U$ and a simple diagram chase of

$$\begin{tikzcd} \Omega^\bullet(U) \arrow[r, two heads, "j^*"] & \Omega^\bullet(D) \arrow[r, hook, two heads, "\xi^*"] & \Omega^\bullet(D) \\ \Omega^\bullet(U/\mathcal{F}|_U) \arrow[u, hook, "\pi_U^*"'] \arrow[r, hook, two heads, "{[j]^*}"'] & \Omega^\bullet(D/\Gamma) \arrow[u, hook, "\pi_D^*"'] \arrow[ur, hook, "\pi_D^*"'] & \end{tikzcd}$$

implies that $\xi^* j^*\omega = j^*\omega$. It follows that $\bar{\partial}\sigma_\Gamma = j^*\omega$ and since σ_Γ is Γ-invariant, we can apply ψ (the inverse of j^*) on each side to get $\bar{\partial}\psi(\sigma_\Gamma) = \omega$, which concludes the proof. $\square$

We will say that a *partition of unity* $(U_i, \rho_i)_{i\in I}$ *is basic* if $\{U_i\}_{i\in I}$ is a basic open cover and $\{\rho_i\}_{i\in I}$ is a set of basic maps.

[14]Recall that we re-defined the Nijenhuis tensor in this context to be over Q, i.e. $N_J : Q \wedge Q \to Q$, c.f. Definition 5

Lemma 15 (see [2], lemma 2.2.)**.** Let $(M, \mathcal{F})$ be a Riemannian foliation. Then every saturated open cover of M admits a basic partition of unity.

One would like to conclude from this lemma that the sheaves $\mathcal{A}_B^{r,s}$ are fine and thus acyclic. However, we must be cautious, as the saturated topology is not Hausdorff, which is typically assumed in the definition of a fine sheaf (as in Godement's and Bredon's treatments of the theory). We can work around this using the fact that the space $M/\mathcal{F}$ is Hausdorff for a Riemannian foliation with compact leaves.

Let $(\mathrm{Cov}(M_{\mathrm{sat}}), \leq)$ be the poset of saturated open covers of M, where $\{U_i\}_{i\in I} \leq \{V_j\}_{j\in J}$ if for each $j \in J$ there is $r(j) \in I$ (a "refinement map") such that $V_j \subset U_{r(j)}$. A sub-poset $X \subset \mathrm{Cov}(M_{\mathrm{sat}})$ is cofinal if for all $\mathcal{U} \in \mathrm{Cov}(X)$ there exists $\mathcal{V} \in X$ such that $\mathcal{U} \leq \mathcal{V}$.

Lemma 16. Let $(M, \mathcal{F})$ be a Riemannian foliation with compact leaves. Locally finite covers are cofinals among the open covers of M_{sat}.

Proof. Since the leaves are compact, $M/\mathcal{F}$ is paracompact (specifically, $M/\mathcal{F}$ is a V-manifold by [18] Proposition 3.7, and in particular second countable and Hausdorff. Combining this with the easy observation that $M/\mathcal{F}$ is locally compact gives paracompactness). Let $\{U_i\}_{i\in I}$ be an open cover of M_{sat}, then $\{\pi(U_i)\}_{i\in I}$ is an open cover of $M/\mathcal{F}$. Since the latter is paracompact, $\{\pi(U_i)\}_{i\in I}$ admits a refinement $\{V_j\}_{j\in J}$ such that $V_j \subset \pi(U_{r(j)})$. Then $\pi^{-1}(V_j) \subset U_{r(j)}$ and thus $\{U_i\}_{i\in I} \leq \{\pi^{-1}(V_j)\}_{j\in J}$. The fact that $\{\pi^{-1}(V_j)\}_{j\in J}$ is locally finite follows directly. If $x \in M$ is such that $\pi(x)$ intersects exactly the opens $V_{i_1}, ..., V_{i_n}$, then x intersects $\pi^{-1}(V_{i_j})$ $(j = 1, ..., n)$ and no other members of the cover. $\square$

We directly obtain:

Corollary 3. Let $(M, \mathcal{F})$ be a Riemannian foliation with compact leaves. For any sheaf $\mathcal{G}$ on M_{sat}, the Cěch cohomology is computed by $\check{H}^i(X, \mathcal{G}) = \varinjlim \check{H}(\mathcal{U}, \mathcal{G})$ where the direct limit is taken over locally finite covers $\mathcal{U}$ of M_{sat}.

Lemma 17. Let $(M, \mathcal{F})$ be a Riemannian foliation with compact leaves. The sheaves $\mathcal{A}_B^k$ over M_{sat} are acyclic. If $(M, \mathcal{F})$ is Hermitian, then $\mathcal{A}_B^{r,s}$ are acyclic over M_{sat}.

Before proving the lemma, recall Cartan's theorem (see Theorem 5.9.2. in Godement's [8]) on how to handle the equivalence of Cěch cohomology and sheaf cohomology over non-paracompact (or non-Hausdorff) spaces[15]

Lemma 18 (Cartan's theorem)**.** Let X be a topological space, $\mathcal{U}$ a basis for the topology of X, and $\mathcal{F}$ a sheaf of abelian groups on X. Suppose that $\mathcal{U}$ is closed under finite intersections. If $\check{H}^i(U, \mathcal{F}) = 0$ for all $i > 0$ and all $U \in \mathcal{U}$, then $\check{H}^i(X, \mathcal{F}) = H^i(X, \mathcal{F})$ for all i.

[15]Caution: in Godement's treatment of the theory (and very often in the French literature), paracompact means that every open cover has a locally finite refinement *and* that the space is Hausdorff. Thus the folklore statement that "sheaf cohomology equals Cěch cohomology over paracompact spaces" does not apply on M_{sat}.

Proof of lemma 17. Let U be any open set in M_{sat}. We first show $\check{H}(U, \mathcal{A}_B^k) = 0$. Since locally finite cover are cofinal in M_{sat}, it is enough to show that $\check{H}^i(\mathcal{U}, \mathcal{A}_B^k) = 0$ for all locally finite covers $\mathcal{U} = \{U_i\}_{i \in I}$ of U. For this, we define a homotopy between the identity and the zero-morphism at the level of cochains. Recall that the Cěch complex of the pair $(\mathcal{U}, \mathcal{A}_B^p)$ is given by $\check{C}^p(\mathcal{U}, \mathcal{A}_B^k) = \prod_{i_1,\ldots,i_p \in I^p} \mathcal{A}_B^p(U_{i_0,\ldots,i_p})$ (where $U_{i_0,\ldots,i_p} = \cap_{j=0}^p U_{i_j}$) with differential $(\delta\omega)_{i_0,\ldots,i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \omega_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}$. Let $\{\psi_i\}_{i \in I}$ be a basic partition of unity with respect to $\{U_i\}_{i \in I}$ and define a contracting homotopy $D : \check{C}^p(\mathcal{U}, \mathcal{A}_B^p) \to \check{C}^p(\mathcal{U}, \mathcal{A}_B^{p-1})$ by

$$(D\omega)_{i_0,\ldots,i_{p-1}} := \textstyle\sum_{j \in I} \psi_i \omega_{i,i_0,\ldots,i_{p-1}}. \tag{19}$$

Observe that $(D\omega)_{i_0,\ldots,i_{p-1}}$ is well defined since the sum in (19) is finite at each point $x \in U$. $(D\omega)_{i_0,\ldots,i_n}$ is smooth since it is a sum of smooth forms and basic since the ψ_i and $\omega_{i_0,\ldots,i_n}$ are basic. The operator D is thus well defined. We claim that $D\delta + \delta D = \mathrm{id}$. Indeed, on one hand,

$$\begin{aligned}(D\delta\omega)_{i_0,\ldots,i_p} &= \textstyle\sum_{i \in I} \left(\psi_i \omega_{i_0,\ldots,i_n} - \sum_{k=0}^p (-1)^k \psi_i \omega_{i,i_0,\ldots,\hat{i_k},\ldots,i_p}\right) \\ &= \omega_{i_0,\ldots,i_p} + \textstyle\sum_{i \in I} \sum_{k=0}^p (-1)^k \psi_i \omega_{k,i_0,\ldots,\hat{i_k},\ldots,i_p}\end{aligned}$$

and on the other hand

$$(\delta D\omega)_{i_0,\ldots,i_p} = \textstyle\sum_{i \in I} \sum_{k=0}^p (-1)^k \psi_i \omega_{i,i_0,\ldots,\hat{i_k},\ldots,i_p}$$

Adding up the two terms yields $(D\delta + \delta D)\omega = \omega$. It follows directly that $\check{H}(\mathcal{U}, \mathcal{A}_B^p) = 0$. Cartan's theorem then implies that $0 = \check{H}^i(M_{\text{sat}}, \mathcal{A}_B^p) = H^i(M_{\text{sat}}, \mathcal{A}_B^p)$ for $i > 0$, which yields acyclicity of $\mathcal{A}_B^p$. The acyclicity of $\mathcal{A}_B^{r,s}$ is proved analogously. □

Remark 11. A slight modification of the proof would show that any sheaf of modules over the ring of basic smooth functions is acyclic over M_{sat}.

Corollary 4 (Transverse Dolbeault Isomorphism)**.** Let (M, g, J) be a compact transversely Kähler manifold such that the foliation induced by $\mathcal{D}_J^\infty$ has compact leaves. Then $H^s(M_{\text{sat}}, \Omega_B^r) = H_B^{r,s}(M)$.

Proof. From Lemma 14, the sequence

$$0 \to \Omega^r \hookrightarrow \mathcal{A}^{r,0} \to \mathcal{A}^{r,1} \to \cdots$$

is exact, and from Lemma 17, the sheaves $\mathcal{A}_B^{r,s}$ are acyclic over M_{sat}. The conclusion follows immediately. □

Remark 12. In the setup of Corollary 4, it also follows from [2] Proposition 2.6 that the sequence of sheaves

$$\cdots \to \mathcal{A}_B^{p-1} \to \mathcal{A}_B^p \to \mathcal{A}_B^{p+1} \to \ldots$$

is exact over M_{sat} (understanding $\mathcal{A}_B^p$ as the sheaf of basic forms over $\mathbb{R}$ or $\mathbb{C}$). Then $H^i(M_{\text{sat}}, \underline{\mathbb{R}}) = H_B^i(M, \mathbb{R})$ and $H^i(M_{\text{sat}}, \underline{\mathbb{C}}) = H_B^i(M, \mathbb{C})$.

Definition 16. Let (M, J) be an almost complex manifold, a *J-bundle* of rank k (or a J-holomorphic vector bundle) is a smooth vector bundle L which admits a trivialization $(U_\alpha, \varphi_\alpha)$ such that the transition maps $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ are (J, i)-holomorphic maps.

Remark 13. Let $\mathcal{E}$ be the sheaf of sections of a smooth vector bundle E over (M, J). The sheaf $\mathcal{E}$ can be seen as a $\mathcal{O}^J_M$-module via the inclusion $\mathcal{O}^J_M \hookrightarrow \mathcal{C}^\infty_{M,\mathbb{C}}$. Then E is a J-vector bundle if and only if $\mathcal{E}$ is a locally free $\mathcal{O}^J_M$-module.

Example 7. Let (M^{2n}, J) be an almost complex manifold. The generalization of the canonical bundle to the almost complex case is generally understood as $K_M := \wedge^n (TM^{1,0}_{\mathbb{C}})^*$. This is not a locally free $\mathcal{O}^J_M$-module as in the complex case. When $\mathcal{D}^\infty_J$ is of constant type, say of type q, another generalization is given by $K_B := \wedge^q (Q^{1,0}_{\mathbb{C}})^*$, where $Q = TM/D^\infty_J$. We claim this is a J-line bundle. By theorem 4, there is a (J, i)-holomorphic submersion $\pi : U \to V \subset \mathbb{C}^q$ such that

$$0 \to \mathcal{D}^\infty_J|_U \to TM|_U \overset{d\pi}{\to} \pi^* T\mathbb{C}^q|_U \to 0 \quad \text{is exact}$$

so that π induces an isomorphism of sheaf over U, $d\pi : \mathcal{Q}^{1,0}|_U \to \pi^*(T\mathbb{C}^q)^{1,0}|_V$ and since $T\mathbb{C}^q \cong \mathcal{O}^{\oplus q}_{\mathbb{C}^q}$, we get

$$Q^{1,0}|_U \cong \pi^* T\mathbb{C}^q|_V \cong \pi^* \mathcal{O}^{\oplus q}_{\mathbb{C}^q}|_V \cong (\mathcal{O}^J_M)^{\oplus q}|_U$$

Where the last isomorphism was proven in theorem 4. To see how their Chern classes relates, considere $0 \to \mathcal{D}^\infty_J \to TM \to TM/\mathcal{D}^\infty_J \to 0$. Complexify the sequence and take the $(1,0)$ part of each bundle. The sequence remains exact, and thus the determinant bundles are related by

$$\wedge^n (TM^{1,0})^* \cong \wedge^p ((\mathcal{D}^\infty_J)^{1,0})^* \otimes \wedge^q (Q^{1,0})^*.$$

so that $c_1(K_M) = c_1(K_B) - c_1(\wedge^p \mathcal{D}^\infty_J)$.

Definition 17. Let (M, J) be an almost complex manifold of constant type. A J-bundle is *basic* if it can be trivialized over saturated open sets (where the saturated topology is induced by $\mathcal{D}^\infty_J$).

A morphism of J-line bundles $f : L_1 \to L_2$ is simply a morphism of line bundles such that the induced map on sections $f_* : \Gamma(L_1|_U) \to \Gamma(L_2|_U)$ is $\mathcal{O}^J_M(U)-$linear for every open set $U \subset M$.

Definition 18. Let (M, J) be an almost complex manifold. Define $\mathrm{Pic}(M)$ to be the group of J-line bundles up to isomorphisms with the group operation given by the tensor product of line bundles. If M is of constant type, define $\mathrm{Pic}_B(M)$ to be the group of basic J-line bundles with tensor product up to isomorphism.

Since $\mathcal{O}^J_M$ is a sheaf of commutative rings, the same proof as in the complex category shows that Pic(M) is an abelian group. As in the complex category, the identification[16] $\check{H}^1(M_{\mathrm{sat}}, \mathcal{O}^{J,*}_M) = H^1(M_{\mathrm{sat}}, \mathcal{O}^{J,*}_M)$ and $\check{H}^1(M, \mathcal{O}^{J,*}_M) = H^1(M, \mathcal{O}^{J,*}_M)$ implies

[16]Recall that for $i = 0, 1$, the i-th Čech Cohomology group and the i-th sheaf cohomology groups agree over any topological space and choice of sheaf (see [8], Corollary to Theorem 5.9.1)

Remark 14. Let (M, J) be an almost complex manifold. Then there is a group isomorphism $\mathrm{Pic}(M) \cong H^1(M, \mathcal{O}_M^{J,*})$. If (M, J) is also of constant type, then $\mathrm{Pic}_B(M) \cong H^1(M_{\mathrm{sat}}, \mathcal{O}_M^{J,*})$. Observe that the canonical map $\mathrm{Pic}_B(M) \hookrightarrow \mathrm{Pic}(M)$ is injective since if $[L]$ is trivial in $\mathrm{Pic}(M)$, then it is also trivial in $\mathrm{Pic}_B(M)$.

Proposition 11. Let (M, J) be an almost complex manifold. Consider the exponential map $\exp : \mathcal{O}_M^J \to \mathcal{O}_M^{J,*}$ defined by taking $f \in \mathcal{O}_M^J(U)$ to $e^{2\pi i f} \in \mathcal{O}_M^{J,*}(U)$, where e denotes the usual holomorphic exponential map $e : \mathbb{C} \to \mathbb{C}\backslash\{0\}$. The following sequence is always exact in the usual topology of M:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_M^J \to \mathcal{O}_M^{J,*} \to 0 \tag{20}$$

Assuming that (M, J) is transversely Kähler and that $\mathcal{D}_J^\infty$ has compact leaves, the sequence (20) is also exact over the saturated topology induced by $\mathcal{D}_J^\infty$.

Proof. The proof in the usual topology is exactly as in the complex case. Over the saturated topology, exactness at $\underline{\mathbb{Z}}$ and $\mathcal{O}_M^J$ are clear. The subtle point is that $\exp : \mathcal{O}_{M_{\mathrm{sat}}}^J \to \mathcal{O}_{M_{\mathrm{sat}}}^{J,*}$ is also surjective in the saturated topology. We show this at the level of stalks. Let $[U, f] \in \mathcal{O}_{M_{sat},x}^{J,*}$ be a germ of J-holomorphic functions over a saturated open set $x \in U \subset M_{\mathrm{sat}}$. Let L be the leaf containing x and let $\epsilon := |f(x)|$. By continuity of f, there exists $\delta > 0$ such that $d(x, y) \leq \delta$ implies $f(y) \in B(f(x), \epsilon) \subset \mathbb{C}\backslash\{0\}$ (here d is the induced metric from the Riemannian metric on M). Note that the latter set is simply connected and does not contain 0. Now, by [18] Lemma 3.7, if $\delta' > 0$ is small enough, the tubular neighbourhood V of L of radius δ', given by the image of $\exp : (\nu TL)_{\delta'} \xrightarrow{\sim} V$, is a saturated open set. Each $D_y = \exp(\nu T_y L_{\delta'})$ is a transversal which intersects each leaf in V (this uses the fact that g is a bundle-like metric and that the leaves are compact). Now since L is compact, we can choose δ' such that $V \subset U$ and $\delta' \leq \delta$. By Theorem 3, f is constant along the leafs of $\mathcal{D}_J^\infty$, and thus $f(V) = f(D_x)$. Observe that $y \in D_x$ implies $d(x, y) < \delta$ and thus $f(y) \in B(f(x), \epsilon)$, so that $f(V) = f(D_x) \subset B(f(x), \epsilon)$. Now let log be a branch of the logarithm on $B(f(x), \epsilon)$, we have that $[V, \log(f)]$ maps to $[V, f]$ under exp, which proves surjectivity.

□

We then have an induced long exact sequence of sheaves which allows to define a notion of first Chern class as in the complex category: given $[L] \in H^1(M, \mathcal{O}_M^{J,*})$, we let $c_1([L]) \in H^2(M, \mathbb{Z})$ be the image of the connecting homomorphism:

$$\cdots \to H^1(M, \mathcal{O}_M^J) \to H^1(M, \mathcal{O}_M^{J,*}) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \to \cdots .$$

We also have a Chern class morphism for saturated J-bundles, via the long exact sequence

$$\cdots \to H^1(M_{\mathrm{sat}}, \mathcal{O}_M^J) \to H^1(M_{\mathrm{sat}}, \mathcal{O}_M^{J,*}) \xrightarrow{c_1} H^2(M_{\mathrm{sat}}, \mathbb{Z}) \to \cdots \tag{21}$$

Theorem 9 (Lefschetz theorem on (1,1)-class)**.** Let (M, J) be a minimal transversaly Kähler manifold such that $\mathcal{D}_J^\infty$ has compact leafs, Then $c_1(\mathrm{Pic}_B(M)) = H^2(M/\mathcal{D}_J^\infty, \mathbb{Z}) \cap H_B^{1,1}(M)$.

Before proving the theorem, considere the following

Lemma 19. Let M be a manifold, let $\mathcal{F}$ be a foliation on M, and let $\pi : M \to M/\mathcal{F}$ be the projection map. If $\mathcal{G}$ is a sheaf of abelian groups on $M/\mathcal{F}$, then

$$H^i(M/\mathcal{F}, \mathcal{G}) \cong H^i(M_{\mathrm{sat}}, \pi^{-1}\mathcal{G}).$$

In particular, $H^i(M/\mathcal{F}, \underline{\mathbb{Z}}) \cong H^i(M_{\mathrm{sat}}, \underline{\mathbb{Z}})$

Proof. By definition, the map $U \mapsto \pi^{-1}(U)$ gives a bijection between opens of M/F and opens of M_{sat}. Hence, for any sheaf $\mathcal{G}$ on M/F and any open $U \subset M/F$,

$$\pi_*\pi^{-1}\mathcal{G}(U) = (\pi^{-1}\mathcal{G})(\pi^{-1}U) = \mathcal{G}(U),$$

and for any sheaf $\mathcal{H}$ on M_{sat} and any open $V \subset M_{\mathrm{sat}}$,

$$\pi_*\pi^{-1}\mathcal{H}(V) = (\pi_*\mathcal{H})(\pi(V)) = \mathcal{H}(V).$$

Thus π^{-1} induces an exact equivalence between the category of abelian sheaves over M_{sat} and the category of abelian sheaves over $M/\mathcal{F}$. Then upon noticing that $\Gamma(M_{\mathrm{sat}}, \pi^{-1}\mathcal{G}) = \Gamma(M/F, \mathcal{G})$ for any sheave $\mathcal{G}$ on $M/\mathcal{F}$, and since π^{-1} is exact and preserves injectives (preservation of injectives follows from π^{-1} having an inverse functor), one has a canonical isomorphism of right derived functors:

$$R^k\Gamma(M_{\mathrm{sat}}, \pi^{-1}\mathcal{G}) \;\cong\; R^k\Gamma(M/F, \mathcal{G}).$$

It is straightforward to see that $\pi^{-1}\underline{\mathbb{Z}}_{M/\mathcal{F}} = \underline{\mathbb{Z}}_{M_{\mathrm{sat}}}$ and the conclusion follows immediatly. $\square$

Proof of theorem 9. The proof follows the lines of the usual proof in the complex case. Note first that the inclusions

$$\begin{tikzcd} \underline{\mathbb{Z}} \arrow[r, hook] \arrow[rrr, hook, bend right] & \underline{\mathbb{R}} \arrow[r, hook] & \underline{\mathbb{C}} \arrow[r, hook] & \mathcal{O}^J_M \end{tikzcd}$$

induce a commutative diagram

$$\begin{tikzcd} H^1(M_{\mathrm{sat}}, \mathcal{O}^{J,*}_M) \arrow[r, "c_1"] & H^2(M_{\mathrm{sat}}, \underline{\mathbb{Z}}) \arrow[r, "i"] \arrow[d, "i"] & H^2(M_{\mathrm{sat}}, \mathcal{O}^J_M) \\ & H^2(M_{\mathrm{sat}}, \underline{\mathbb{R}}) \arrow[d, "j"] & \\ & H^2(M_{\mathrm{sat}}, \underline{\mathbb{C}}) \arrow[ruu, "\pi^{0,2}"'] & \end{tikzcd} \tag{22}$$

In order to bookkeep the morphism $\pi^{0,2}$, we go back to our basic resolutions of $\mathcal{O}^J_M$, $\underline{\mathbb{R}}$ and $\underline{\mathbb{C}}$. We have the following commutative diagram with exact rows (recall Remark 12 and Lemma 14):

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}^J_M \arrow[r, hook] & \mathcal{A}^{0,0}_B \arrow[r, "\bar\partial"] & \mathcal{A}^{0,1}_B \arrow[r, "\bar\partial"] & \mathcal{A}^{0,2}_B \arrow[r, "\bar\partial"] & \cdots \\ 0 \arrow[r] & \underline{\mathbb{C}} \arrow[r, hook] \arrow[u] & \mathcal{A}^0_B \arrow[r, "d"] \arrow[u, "\mathrm{id}"] & \mathcal{A}^1_B \arrow[r, "d"] \arrow[u, "\pi^{0,1}"] & \mathcal{A}^2_B \arrow[r, "d"] \arrow[u, "\pi^{0,2}"] & \cdots \\ 0 \arrow[r] & \underline{\mathbb{R}} \arrow[r, hook] \arrow[u] & \mathcal{A}^0_B \arrow[r, "d"] \arrow[u, hook, "i"] & \mathcal{A}^1_B \arrow[r, "d"] \arrow[u, hook, "i"] & \mathcal{A}^2_B \arrow[r, "d"] \arrow[u, hook, "i"] & \cdots \end{tikzcd} \tag{23}$$

The map j is then given by $j([\omega]) = [\omega]$ and $\pi^{0,2}$ is given by $\pi^{0,2}([\omega]) = [\pi^{0,2}(\omega)]$ (where $\pi^{0,2}$ is the projection of 2−forms to $(0,2)$-forms in (23). We tacitly identify $H^2(M_{\text{sat}}, \underline{\mathbb{R}}) = H^2_B(M, \mathbb{R})$, the basic cohomology over real-valued forms and $H^2(M_{\text{sat}}, \underline{\mathbb{R}}) = H^2_B(M, \mathbb{C})$, the basic cohomology over complex valued forms.) Let $\sigma \in H^1(M_{\text{sat}}, \mathcal{O}^{J,*}_M)$, and let $c_1(\sigma) = [\omega] \in H^2_B(M, \mathbb{R})$ where ω is a (real) Δ_B-harmonic representative of $[\omega]$ (such representative exists by Theorem 6). Then $c_1(\sigma)$ seen as a class in $H^2_B(M, \mathbb{C})$ is given by $j[\omega] = [\omega]$. Since (21) is exact, $i \circ c_1(\sigma) = 0$ and by commutativity of (22) this implies $[\pi^{0,2}\omega] = 0$. Since ω is Δ_{B}-harmonic, $\pi^{0,2}\omega$ is $\Delta_{\bar{\partial}_B}$−harmonic (by Proposition6) so that $\pi^{0,2}\omega = 0$. Since ω is a real-form, we also get that $\pi^{2,0}\omega = 0$, and thus $[\omega] \in H^{1,1}_B$ and $c_1(\sigma) \in H^2(M_{\text{sat}}, \mathbb{Z}) \cap H^{1,1}_B$. This gives the inclusion $\operatorname{Im} c_1 \subset H^{1,1}_B(M) \cap H_2(M_{\text{sat}}, \mathbb{Z})$.

For the other inclusion, suppose that $\sigma \in H^{1,1}_B(M) \cap H^2(M_{\text{sat}}, \mathbb{Z})$. Then $i(\sigma) = 0$ in $H^2(M_{\text{sat}}, \mathcal{O}^J_M)$ so that $\sigma \in \operatorname{Im}(c_1)$ by exactness. The result then follows from the canonical identification of Lemma 19: $H^\bullet(M/\mathcal{D}^\infty_J, \underline{\mathbb{Z}}) \cong H^\bullet(M_{\text{sat}}, \underline{\mathbb{Z}})$.

□

Example 8 (Kodaira-Thurston torus, continued)**.** Recall from Example 4 that the leaf space $KT/\mathcal{D}^\infty_J$ is given by a smooth submersion $\pi : KT \to T^2 := \mathbb{R}^2/\mathbb{Z}^2$. Since the leaf space is a smooth manifold, it is not hard to give an explicit generator of $H^{1,1}_B(KT) \cap H^2(KT/\mathcal{D}^\infty_J, \mathbb{Z})$. Let J_0 be the standard complex structure on T^2, i.e. if $e_1 = (1,0)$ and $e_2 = (0,1)$, we let $J_0 e_1 := -e_2$. Then π is a (J, J_0)-holomorphic map. Indeed, keeping the notations of Example 4 we see from (16), that $d\pi(X_1) = (1,0)$ and $d\pi(X_4) = (0,1)$, a quick computation then shows $d\pi \circ J = J_0 \circ d\pi$. Let $\mathcal{O}_{T^2}$ be the sheaf of holomorphic functions on T^2. The natural map $\pi^\# : \pi^{-1}\mathcal{O}_{T^2} \to \mathcal{O}^J_{\text{KT}}$ induced by $\pi : \text{KT} \to T^2$ is in fact an isomorphism by Theorem 4. It then follows from Lemma 19 that

$$\operatorname{Pic}_B(KT) = H^1(KT_{\text{sat}}, \mathcal{O}^{J,*}_{\text{KT}}) \cong H^1(T^2, \mathcal{O}^*_{T^2}) = \operatorname{Pic}(T^2).$$

Let [pt] be the line bundle on T^2 corresponding to the divisor given by some point. Recall that $c_1([\text{pt}])$ generates $c_1(\operatorname{Pic}(T^2))$. Then the pullback line bundle $L := \pi^*[\text{pt}]$ generates $c_1(\operatorname{Pic}_B(M)) = H^{1,1}_B(KT) \cap H^2(T^2, \mathbb{Z})$. Lastly, note that since T^2 is smooth, $\pi^{-1}\mathcal{A}^{r,s}_{T^2} = \mathcal{A}^{r,s}_B$. Thus, from Lemma 19 and the transverse Dolbeault isomorphism 4, we get

$$H^{r,s}(T^2) = H^s(T^2, \Omega^r) \cong H^s(\text{KT}_{\text{sat}}, \Omega^r_B) = H^{r,s}_B(KT)$$

Thus the Hodge diamond of (KT, J) is just the Hodge diamond of its leaf space T^2. We know from 9 that $H^{r,s}_B(KT)$ is a subspace of Cirici-Wilson's Dolbeault cohomology. This brings us back to question 4 of the introduction. According to [5] Example 5.16, $h^{1,1}_{\text{Dolb}}(KT) \geq 4$. We know from Proposition 9 that $H^{1,1}_B(KT) \hookrightarrow H^{1,1}_{\text{Dol}}(KT)$ so there are at least three generators of $H^{1,1}_{\text{Dol}}(KT)$ for which we don't have a geometric interpretation.

We give two proof of lemma 7:

Lemma (Unique continuation)**.** Let $f \in \mathcal{O}_M^J(U)$ where $U \subset M$ is a connected open subset and suppose that f vanishes to infinite order at a point $p \in U$. Then $f \equiv 0$ on U.

Let $B_\varepsilon(p)$ be a ball in $\mathbb{R}^n$, recall that a locally integrable function $f : B_\varepsilon(p) \to \mathbb{C}$ vanishes to infinite order at p if

$$\int_{|x|\leq r} |f(x)| = O(r^k) \quad \text{for every } k \in \mathbb{N}.$$

If f is smooth this is equivalent to ask that the ∞-jet of f vanishes at zero. In partilcular, the notion is well defined for smooth functions on a manifold.

Begin by recalling that a $J-$**holomorphic disk** on an almost complex manifold (M, J) is a smooth map $u : \Delta \to M$ from the open unit disk $\Delta \subset \mathbb{C}$ to M which satisfies the differential relation $du \circ i = J \circ du$. We have the following theorem[17]:

Theorem (J.P. Rosay, S. Ivashkovich [6], prop 1.2.2). Let (M, J) be an almost complex manifold and $p \in M$ For every sufficiently small $v \in T_pM$, there exists a smooth J-holomorphic map $u_{p,v}$ from Δ into M such that $u_{p,v}(0) = p$ and $(du_{p,v})_0(\partial_x) = v$ (Here, ∂_x and ∂_y form the standard basis of $T_0\Delta = \mathbb{R}^2$). Moreover, $u_{p,v}$ can be chosen with C^1 dependence on the parameters (p, v) in TM. //

Corollary. Let (M, J) be an almost complex manifold and $p \in M$. There exists an open neighburhood U of p which is covered by the image of $J-$holomorphic disks u satisfying $u(0) = p$. //

Proof. The theorem above implies that there is an open neighbourhood $V \subset T_pM$ of $0 \in T_pM$ for which each $v \in V$ has a $J-$holomorphic disk $u_{p,v}$ with parameter v being $\mathcal{C}^1$.Considere the following "exponential map" of class C^1:

$$\Phi : \Delta \times V \to M$$
$$(x, v) \mapsto u_{v,p}(x)$$

this is clearly a submmersion at $(0, 0) \in \Delta \times V$. Indeed, given $w \in T_pM$, there exists $v \in V$ such that $\lambda v = w$. Then evaluating $d\Phi_{(0,0)}$ on the path $\gamma(t) = ((\lambda t, 0), v) \in \Delta \times V$ yield

$$(d\Phi)_{(0,0)}(\gamma'(0)) = d(u_{p,v})_0(\lambda\partial_x) = w.$$

It follows immediately that the image of Φ covers an open neighbourhood of $\Phi(0, 0) = p$, which is exactly to say that this neighbourhood is covered by J-holomorphic disks. □

The proof of lemma 7 follows immediately :

[17] In fact, Rosay and Ivashkovich prove the existence of J-holomorphic disk with prescribed k-th order jet at a point with C^1 dependence on parameters for any k as long as J is smooth enough.

Proof of lemma 7. Let U be a connected open subset of M and let $f : U \to \mathbb{C}$ be a $J-$holomorphic function vanishing to infinite order at $p \in M$. Let $p \in V \subset U$ be an open subset covered by J-holomorphic disks u with $u(0) = p$. Let $q \in V$ be arbitrary and $u : \Delta \to V$ be a J-holomorphic disk passing trough q. Since u and f are both J-holomorphic, $f \circ u$ is in fact holomorphic (notice the relation $df \circ du\, i = df \circ J \circ du = i\, df \circ du$). Then $f \circ u$ vanishes to infinite order at 0, so it vanishes everywhere on Δ by the usual unique continuation theorem for holomorphic functions in one variable. In particular, $f(q) = 0$. This gives $f|_V \equiv 0$ and thus the subset of U on which f vanishes to infinite order is open. But it is also closed by continuity of all the partial derivatives of f. Hence the connectedness of U implies $f|_U \equiv 0$. □

Remark 15. Note that contrary to the i-holomorphic case, the unique continuation property does not hold for the partial derivatives of a J-holomorphic function in general. In fact, it need not hold even if J is integrable, as the next example shows.

Example 9. On a neighbourhood of $0 \in \mathbb{C}^2$, considere the almost complex structure J defined by asking that the following vectorfields be of type (0,1):

$$\begin{aligned} X_1 &= \partial_{\bar{z}_1} \\ X_2 &= \partial_{\bar{z}_2} + e^{-1/Re(z_2)^2} \partial_{z_1} \end{aligned} \tag{24}$$

Since $[X_1, X_2] = 0$, J is integrable. Let $\Phi = \Phi(z_2, \bar{z}_2)$ be a solution to the inhomogenous equation $\partial_{\bar{z}_2} \Phi = e^{-1/Re(z_2)^2}$ in a neighbourhood of the origin[18]. Clearly, $u(z_1, z_2) := z_1 - \Phi(z_2, \bar{z}_2)$ is a smooth solution to the system, but $\partial_{\bar{z}_2} u$ vanishes to infinite order at 0 and is not everywhere equal to 0.

Remark 16. One should also notes that any J-holomorphic function f satisfies the maximum principle (if $f(p)$ is a local maximum or minimum, then f is constant in a neighbourhood of p). This is seen directly from [5], corollary 4.9.

We give an other proof of the unique continuation property which does not relies on the theory of J-holomorphic curve. The strategy is to show that any regular enough function $u : U \to \mathbb{C}$ satisfying the system of linear first order PDE

$$\partial_{\bar{z}_j} u + \sum_{i=1}^{n} a_j^i \, \partial_{z_i} u = 0 \quad j = 1, \ldots, n \tag{25}$$

(where the a_i^j's are smooth and $a_i^j(0) = 0$) is such that it's real and imaginary part also satisfies a system of linear second order elliptic PDE (with real coefficients) in a neighbourhood of 0 and then appeal to the classical Aronszajn's theorem :

Theorem (Aronszajn [1], theorem 1, remark 3 and addendum[19].)**.** Let A be a linear elliptic second order partial differential operator on an connected

[18] One uses Cauchy's integral formula (for smooth, complex valued functions) to prove existence of solutions. See [20], p.394 and reference therein.

[19] Theorem 1 states the result for one function $u : U \to \mathbb{R}$, remark 3 extends this to a familty of function $u_1, ..., u_k$ and the Addendum states that it suffice to assume $\partial_{\mu\lambda} a^{ij}$ of class C^2 (instead of $C^{2,1}$, but we dont need this reduction in our context)

open set $U \subset \mathbb{R}^n$

$$A = \sum_{i,j} a^{ij} \partial_{x_i x_j} + \sum_j b^j \partial_{x_j} + a$$

such that the coefficients a^{ij} are of class $C^2(U, \mathbb{R})$ and all other coefficents b^j and a are uniformly bounded. The elliptic requirement means that (a_{ij}) is a symmetric[20] positive definite matrix everywhere on U. Let $u_k : U \to \mathbb{R}$ $k = 1, \ldots, m$, be a family of twice differentiable function such that their derivatives belongs to $L^2_{loc}(U)$. If there exists $M > 0$ such that

$$|Au_k|^2 \leq M \sum_{l=1}^{m} \left(\sum_{i=1}^{n} |\partial_{x_i} u_l|^2 + |u_l|^2 \right) \quad \text{for} \quad k = 1, \ldots, m$$

and each u_k vanishes to infinite order at $p \in U$, then each u_k is equal to 0 on U.

Let us fix some more notations. Recall that the *principal symbol* of a PDO with real coefficients $D = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ given in the real basis ∂_{x_i}, $i = 1, \ldots, n$ is the function $\sigma_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha$, where $\xi = (\xi_{x_1}, ..., \xi_{x_n})$. More generally, if the PDO has complex coefficients, we may expressed it in the basis ∂_{z_i} ,$\partial_{\bar{z}_i}$. We then define the symbol by substituting ∂_{z_i} and $\partial_{\bar{z}_i}$ by $\xi_i = \frac{1}{2}(\xi_{x_i} + i\xi_{y_i})$ and $\bar{\xi_i} = \frac{1}{2}(\xi_{x_i} - i\xi_{y_i})$, respectively. Notice that if the principal part of D has real coefficients, then the "complex" and "real" definitions of the principal symbol agrees. The complex definition is useful to check ellipticity because of the basic relation $\xi_i \bar{\xi_i} = |\xi_i|^2$. Notice that if D is a second order linear PDO, requiring ellipticity is equivalent to asking that $\sigma_m(x, \xi) > 0$ for all $\xi \neq 0$. Notice also that the principal symbol of the product of two PDOs is equal to the product of their principal symbol[21] . Lastly, if $D = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ is a PDO with complex coefficients, define its conjugate $\bar{D} := \sum_{|\alpha| \leq m} \bar{a}_\alpha \bar{\partial}^\alpha$ ($\partial_{\bar{z}_i}$ is sent to ∂_{z_i}).

Second proof of lemma 7. Let $L_j := \partial_{\bar{z}_j} + \sum_{i=1}^n a^i_j \partial_{z_i}$, where each a^j_i is smooth and $a^j_i(0) = 0$. We observed that a J-holomorphic function u is a solution to $L_j u = 0$ for each $j = 1, \ldots, n$. Define $L := \sum_{i=1}^n \bar{L}_j L_j$. The idea of studying L comes from the observation that it can be seen as a small perturbation of the usual Laplacian, which is obtained by setting $L_j = \partial_{\bar{z}_i}$. Clearly, $Lu = 0$. We claim that L is a second order linear elliptic *PDO* in some neighbourhood of 0, with *real* coefficients of order two. We start by showing that the second order part of L is real:

$$\begin{aligned}
\sum_j \bar{L}_j L_j &= \sum_j (\partial_{z_j} + \sum_i \bar{a}^i_j \partial_{\bar{z}_i})(\partial_{\bar{z}_j} + \sum_i a^i_j \partial_{z_i}) \\
&= \sum_j \left(\partial_{z_j \bar{z}_j} + \sum_i \bar{a}^i_j \partial_{z_i \bar{z}_j} + \sum_i a^i_j \partial_{z_j \bar{z}_i} + \sum_i \sum_k \bar{a}^i_j a^k_j \partial_{\bar{z}_i z_k} \right) + \text{l.o.t.} \\
&= \sum_j \left(\Delta_j + 2\text{Re}(\sum \bar{a}^i_j \partial_{z_i \bar{z}_j}) + 2\text{Re}(\sum_{i \leq k} \bar{a}^i_j a^k_j \partial_{\bar{z}_i z_k}) \right) + \text{l.o.t.}
\end{aligned}$$

[20](One can always manipulate the coefficients to get symmetry; ellipticity is about postive definiteness)

[21]check Shubin's invitation to PDE [25] cor. 1.2. if this statement is unclear.

Where $\Delta_j = \partial_{x_j}^2 + \partial_{y_j}^2$ is the Laplacian. The last expression is obtained by noticing that appart from $\partial_{z_j\bar{z}_j} = \Delta_j$, each term in the second equality comes with its conjugate, and for any complex PDO, we have $D + \bar{D} = 2\mathrm{Re}(D)$. Next, we compute the principal symbol of L:

$$\begin{aligned}
\sigma_2(L)(x,\xi) &= \sum_j \sigma_1(\bar{L}_i)\sigma_1(L_i)(x,\xi) \\
&= \sum_j (\xi_j + \sum_i \bar{a}_j^i(x)\bar{\xi}_i)(\bar{\xi}_j + \sum_i a_j^i(x)\xi_i) \\
&= \sum_j |\bar{\xi}_j + \sum_i a_j^i(x)\xi_i|^2.
\end{aligned}$$

So at least $\sigma_2(L)(x,\xi) \geq 0$. To see that it is strictly greater than 0 on a punctured neighbourhood of 0, we observe that $\sigma_2(L)(x,\xi) = 0$ is equivalent to

$$-\bar{\xi}_j = \sum_i a_j^i(x)\xi_i \quad \text{for} \quad i = 1, \dots, n.$$

Let $\xi = (\xi_1, \dots, \xi_n)$ and $A = (a_j^i)$, the equality aboves writes $A\xi = -\bar{\xi}$. Using the triangular inequality we see that

$$|A||\xi| \geq |A\xi| = |\xi|$$

So it is necessary that either $\xi = 0$ or $|A| \geq 1$. Since A is continous and $A(0) = 0$, we can choose a neighbourhood V of 0 on which $|A| < 1$, and thus L is elliptic on V. We now set ourself up to apply Aronsajn's theorem. We let P_2 be the second order part of L and $P_{\leq 1}$ the lower order terms. Given u a solution to $L_i u = 0$ for each i, we set $u = v + iw$. Since $Lu = 0$, we have that

$$\begin{aligned}
0 &= \mathrm{Re}(Lu) \\
&= \mathrm{Re}(P_2 u) + \mathrm{Re}(P_{\leq 1} u) \\
&= P_2 v + \mathrm{Re}(P_{\leq 1})v - \mathrm{Im}(P_{\leq 1})w \qquad (\dagger)
\end{aligned}$$

and similarly,

$$0 = \mathrm{Im}(Lu) = P_2 w + \mathrm{Re}(P_{\leq 1})w + \mathrm{Im}(P_{\leq 1})v. \qquad (\dagger\dagger)$$

Since the a_i^j are smooth functions, the coefficients of the real PDOs $\mathrm{Re}(P_{\leq 1})$ and $\mathrm{Im}(P_{\leq 1})$ are smooth as well, so upon choosing a smaller neighbourhoud of 0, which we again name V, we can assume that the coefficients of $\mathrm{Re}(P_{\leq 1})$ and $\mathrm{Im}(P_{\leq 1})$ are uniformly bounded by some constant M. Then ($\dagger$) implies

$$|P_2 v|^2 = |\mathrm{Re}(P_{\leq 1})v - \mathrm{Im}(P_{\leq 1})w|^2 \leq M^2 \left(\sum_{i=1}^{2n} |\partial_i v|^2 + |v|^2 + |\partial_i w|^2 + |w|^2 \right)$$

The equality ($\dagger\dagger$) implies that $|P_2 w|^2$ is bounded by the same expression, so Aronsajn's theorem guarantee the unique continuation property of u on V. The same local to global argument as in the first proof of lemma 7 gives the full statement. $\square$

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